Combinatorics of crystal graphs and Kostka-Foulkes polynomials for the root systems B_n, C_n and D_n.

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Abstract

We use Kashiwara-Nakashima's combinatorics of crystal graphs associated to the roots sytems B_n and D_n to extend the results of [15] and [20] by showing that Morris type recurrence formulas also exist for the orthogonal root systems. We derive from these formulas a statistic on Kashiwara-Nakashima's tableaux of types B_n, C_n and D_n generalizing Lascoux-Schützenberger's charge and from which it is possible to compute the Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$ with restrictive conditions on (λ, μ) . This statistic is different from that obtained in [15] from the cyclage graph structure on tableaux of type C_n. We show that such a structure also exists for the tableaux of types B_n and D_n but can not be simply related to the Kostka-Foulkes polynomials. Finally we give explicit formulas for $K_{\lambda,\mu}(q)$ when $|\lambda| \leq 3$, or $n = 2$ and $\mu = 0$.

1 Introduction

The multiplicity $K_{\lambda,\mu}$ of the weight μ in the irreducible finite dimensional representation $V(\lambda)$ of the simple Lie algebra g can be written in terms of the ordinary Kostant's partition function $\mathcal{P}$ defined from the equality:

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1-x^{\alpha})} = \sum_{\beta} \mathcal{P}(\beta) x^{\beta}$$

where β runs on the set of nonnegative integral combinations of positive roots of g. Thus $\mathcal{P}(\beta)$ is the number of ways the weight β can be expressed as a sum of positive roots. Then we have

$$K_{\lambda,\mu} = \sum_{\sigma \in W} (-1)^{l(\sigma)} \mathcal{P}(\sigma(\lambda+\rho) - (\mu+\rho))$$

where W is the Weyl group of g.
There exists a q-analogue $K_{\lambda,\mu}(q)$ of $K_{\lambda,\mu}$ obtained by substituting the ordinary Kostant's partition function $\mathcal{P}$ by its q-analogue $\mathcal{P}_q$ satisfying

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1-qx^{\alpha})} = \sum_{\beta} \mathcal{P}_q(\beta) x^{\beta}.$$

So we have

$$K_{\lambda,\mu}(q) = \sum_{\sigma \in W} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda+\rho) - (\mu+\rho)).$$

As shown by Lusztig [18] $K_{\lambda,\mu}(q)$ is a polynomial in q with non negative integer coefficients.
For type A_{n-1} the positivity of the Kostka-Foulkes Polynomials can also be proved by a purely combinatorial method. Recall that for any partitions λ and μ with n parts the number of semi-standard tableaux of shape λ and weight μ is equal to the multiplicity of the weight μ in $V(\lambda)$.

In [12], Lascoux and Schützenberger have introduced a beautiful statistic ch_A on dominant evaluation words w that is on words $w = x_1 \cdots x_l$ whose letters x_i are positive integers such that for any $i \geq 1$ with i a letter of w, w contains more letters i than letters $i+1$. Recall that the plactic monoid is the quotient set of the free monoid on the positive integers by Knuth's relations

$$abx \equiv \begin{cases} bax \text{ if } a < x \leq b \\ axb \text{ if } x \leq a < b \end{cases}.$$

The statistic ch_A is the unique function from dominant evaluation words to non-negative integers such that

$$\begin{cases} \mathrm{ch}_A(\emptyset) = 0 \\ \mathrm{ch}_A(xu) = \mathrm{ch}_A(ux) + 1 \text{ if } x \text{ is not the lowest letter of } w \\ \mathrm{ch}_A(xu) = \mathrm{ch}_A(u) \text{ if } x \text{ is the lowest letter of } w \\ \mathrm{ch}_A(\sigma w) = \mathrm{ch}_A(w) \text{ for any } \sigma \in \mathcal{S}_n \\ \mathrm{ch}_A(w_1) = \mathrm{ch}_A(w_2) \text{ if } w_1 \equiv w_2 \end{cases} \tag{1}$$

[8]. Then the charge of the semi-standard tableau T of dominant weight verifies $\mathrm{ch}_A(T) = \mathrm{ch}_A(\mathrm{w}(T))$ where $\mathrm{w}(T)$ is the word obtained by column reading the letters of T from top to bottom and right to left. Lascoux and Schützenberger have proved the equality

$$K_{\lambda,\mu}(q) = \sum_T q^{\mathrm{ch}_A(T)} \tag{2}$$

where T runs on the set of semi-standard tableaux of shape λ and weight μ. The proof of (2) is based on Morris recurrence formula which permits to express each Kostka-Foulkes polynomials related to the root system A_n in terms of Kostka-Foulkes polynomials related to the root system A_{n-1}.
The compatibility of the charge with plactic relations provides alternative ways to compute $\mathrm{ch}_A(T)$. By applying the reverse bumping algorithm on the boxes contained in the longest row of T we obtain a pair (R, T') with R a row tableau whose length is equal to the longest row of T and T' a semi-standard tableau which does not contain the lowest letter t of T such that $\mathrm{w}(T) \equiv \mathrm{w}(R) \otimes \mathrm{w}(T')$. Let R' be the row tableau obtained by erasing all the letters t in R. Then the catabolism of T is the unique semi-standard tableau $\mathrm{cat}(T)$ such that $\mathrm{w}(\mathrm{cat}(T)) \equiv \mathrm{w}(T') \otimes \mathrm{w}(R')$ computed via the bumping algorithm. We have

$$\mathrm{ch}_A(\mathrm{cat}(T)) = \mathrm{ch}_A(T) + r'$$

where r' is the length of R'. Since the number of boxes of $\mathrm{cat}(T)$ is strictly less than that of T, $\mathrm{ch}_A(T)$ can be obtained from T by computing successive catabolism operations. In fact this is this characterization of the charge which is needed to prove (2).
The charge may also be obtained by endowing $ST(\mu)$ the set of semi-standard tableaux of weight μ with a structure of graph. We draw an arrow $T \rightarrow T'$ between the two tableaux T and T' of $ST(\mu)$, if and only if there exists a word u and a letter y which is not the lowest letter of T such that $\mathrm{w}(T) \equiv xu$ and $\mathrm{w}(T') \equiv ux$. Then we say that T' is a cocyclage of T. The essential tool to define this graph structure is yet the bumping algorithm for the semi-standard tableaux. The cyclage graph $ST(\mu)$ contains a unique row tableau L_μ which can not be obtained as the cocyclage of another tableau of $ST(\mu)$. Let T_μ be the unique semi-standard tableau of shape μ belonging to $ST(\mu)$. Then there is no cocyclage of T_μ. For any $T \in ST(\mu)$ all the paths joining L_μ to T have the same length. This length is called the cocharge of T and denoted $\mathrm{coch}_A(T)$. Similarly, all the paths joining T to T_μ have the same length which is equal to the charge of T. The maximal value of ch_A is $\|\mu\| = \mathrm{ch}_A(L_\lambda) = \sum_i (i-1)\mu_i$. Moreover the charge and the cocharge are related by the equality $\mathrm{ch}_A(T) = \|\mu\| - \mathrm{coch}_A(T)$ for any $T \in ST(\mu)$.

Analogues of semi-standard tableaux also exit for the other classical root systems. They have been introduced by Kashiwara and Nakashima [9] via crystal bases theory. For each classical root system these tableaux naturally label the vertices of the crystal graph $B(\lambda)$ associated to the dominant weight λ.
In [15] we have proved that an analogue of Morris recurrence formula exists for the root system C_n. Moreover it is also possible to endow the corresponding set of tableaux with a structure of cyclage graph. From these graphs we have introduced a natural statistic on Kashiwara-Nakashima's tableaux of type C_n and have conjectured that this statistic yields an analogue of Lascoux-Schützenberger's theorem.

This article is an attempt to look at possible generalizations and extensions of these results to the orthogonal roots systems. We establish Morris type recurrence formula for the root systems B_n and D_n. Moreover we show that is possible to endow the set of tableaux of types B_n and D_n with a structure of cyclage graph. Nevertheless the situation is more complicated than for the root system C_n and we are not able to deduce from these graphs a natural statistic relevant for computing the Kostka-Foulkes polynomials. To overcome this problem we change our strategy and define a new statistic χ_n on tableaux of types B_n, C_n and D_n based on the catabolism

operation. Then we prove that this statistic can be used to compute the Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$ with restrictive conditions on (λ,μ). Note that the analogue of (2) with χ_n is false in general. In particular χ_n is not equal to the statistic defined in [15] for the tableaux of type C_n even if the two statistics can be regarded as generalizations of ch_A since they coincide on semi-standard tableaux.

In Section 1 we recall the Background on Kostka-Foulkes polynomials and combinatorics of crystal graphs that we need in the sequel. We also summarize the basic properties of the insertion algorithms and plactic monoids for the root systems B_n, C_n and D_n introduced in [13] and [14]. Section 2 is devoted to Morris type recurrence formulas for types B_n and C_n. In Section 3 we define the catabolism operation for the tableaux of type B_n, C_n and D_n. Then we introduce the statistics χ_n^B, χ_n^C and χ_n^D and prove that analogues of (2) hold for these statistics if λ and μ satisfy restrictive conditions. We also introduce the cyclage graph structure on tableaux of types B_n and D_n and show that a charge statistic related to Kostka-Foulkes polynomials can not be obtained in a similar way that in [15]. Finally we give in Section 4 explicit simple formulas for the Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$ when $|\lambda| \leq 3$, or $n=2$ and $\mu=0$ deduced from the results of Sections 2 and 3.

Notation: In the sequel we frequently define similar objects for the root systems B_n C_n and D_n. When they are related to type B_n (resp. C_n, D_n), we implicitly attach to them the label B (resp. the labels C, D). To avoid cumbersome repetitions, we sometimes omit the labels B, C and D when our definitions or statements are identical for the three root systems.

2 Background

2.1 Kostka-Foulkes polynomials associated to a root system

Let g be a simple Lie algebra and α_i, $i \in I$ its simple roots. Write Q^+ and R^+ for the set of nonnegative integral combinations of positive roots and for the set of positive roots of g. Denote respectively by P and P^+ its weight lattice and its cone of dominant weights. Let $\{s_i, i \in I\}$ be a set of generators of the Weyl group W and l the corresponding length function.
The q-analogue $\mathcal{P}_q$ of the Kostant function partition is such that

$$\prod_{\alpha\in R^+} \frac{1}{1-qx^\alpha} = \sum_{\beta\in Q^+} \mathcal{P}_q(\beta)x^\beta \text{ and } \mathcal{P}_q(\beta)=0 \text{ if } \beta\notin Q^+.$$

Definition 2.1.1 *Let $\lambda,\mu \in P^+$. The Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ is defined by*

$$K_{\lambda,\mu}(q) = \sum_{\sigma\in W} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda+\rho)-(\mu+\rho)).$$

where ρ is the half sum of positive roots.

Let $\beta \in P$. We set

$$a_\beta = \sum_{\sigma\in W} (-1)^{l(\sigma)} (\sigma\cdot x^\beta)$$

where $\sigma \cdot x^\mu = x^{\sigma(\mu)}$. The Schur function s_β is defined by

$$s_\beta = \frac{a_{\beta+\rho}}{a_\rho}.$$

When $\lambda \in P^+$, s_λ is the Weyl character of $V(\lambda)$ the finite dimensional irreducible g-module with highest weight λ. For any $\sigma \in W$, the dot action of σ on $\beta \in P$ is defined by $\sigma \circ \beta = \sigma\cdot(\beta+\rho)-\rho$. We have the following straightening law for the Schur functions. For any $\beta \in P$, $s_\beta = 0$ or there exists a unique $\lambda \in P^+$ such that $s_\beta = (-1)^{l(\sigma)} s_\lambda$ with $\sigma \in W$ and $\lambda = \sigma\circ\beta$. Set $\mathbb{K} = \mathbb{Z}[q,q^{-1}]$ and write $\mathbb{K}[P]$ for the $\mathbb{K}$-module generated by the x^β, $\beta \in P$. Set $\mathbb{K}[P]^W = \{f \in \mathbb{K}[P], \sigma\cdot f = f \text{ for any } \sigma \in W\}$. Then $\{s_\lambda\}$ is a basis of $\mathbb{K}[P]^W$.

To each positive root α, we associate the raising operator $R_\alpha : P \to P$ defined by

$$R_\alpha(\beta) = \alpha + \beta.$$

Given $\alpha_1, ..., \alpha_p$ positive roots and $\beta \in P$, we set $(R_{\alpha_1} \cdots R_{\alpha_p})s_\beta = s_{R_{\alpha_1} \cdots R_{\alpha_p}(\beta)}$. For all $\beta \in P$, we define the Hall-Littelwood polynomial Q_β by

$$Q_\beta = \left(\prod_{\alpha \in R^+} \frac{1}{1 - qR_\alpha} \right) s_\beta$$

where $\dfrac{1}{1 - qR_\alpha} = \sum_{k=0}^{+\infty} q^k R_\alpha^k$.

Theorem 2.1.2 *[19]For any $\lambda, \mu \in P^+$, $K_{\lambda,\mu}(q)$ is the coefficient of s_λ in Q_μ that is,*

$$Q_\mu = \sum_{\lambda \in P^+} K_{\lambda,\mu}(q) s_\lambda.$$

2.2 Kostka-Foulkes polynomials for the root systems B_n, C_n and D_n

We choose to label respectively the Dynkin diagrams of so_{2n+1}, sp_{2n} and so_{2n} by

$$\overset{0}{\circ} \Longleftarrow \overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \overset{4}{\circ} - \cdots \overset{n-1}{\circ}\,,\; \overset{0}{\circ} \Longrightarrow \overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \overset{4}{\circ} - \cdots \overset{n-1}{\circ} \text{ and } \begin{array}{l} \overset{1}{\circ} \\ \quad\diagdown \\ \qquad \overset{2}{\circ} - \overset{3}{\circ} - \overset{4}{\circ} - \cdots \overset{n-2}{\circ} - \overset{n-1}{\circ}. \\ \quad\diagup \\ \underset{0}{\circ} \end{array} \qquad (3)$$

The weight lattices for the root systems B_n, C_n and D_n can be identified with $P_n = \mathbb{Z}^n$ equipped with the orthonormal basis $\varepsilon_{\overline{i}}$, $i = 1, ..., n$. We take for the simple roots

$$\left\{ \begin{array}{l} \alpha_0^{B_n} = \varepsilon_{\overline{1}} \text{ and } \alpha_i^{B_n} = \varepsilon_{\overline{i+1}} - \varepsilon_{\overline{i}},\ i = 1, ..., n-1 \text{ for the root system } B_n \\ \alpha_0^{C_n} = 2\varepsilon_{\overline{1}} \text{ and } \alpha_i^{C_n} = \varepsilon_{\overline{i+1}} - \varepsilon_{\overline{i}},\ i = 1, ..., n-1 \text{ for the root system } C_n \\ \alpha_0^{D_n} = \varepsilon_{\overline{1}} + \varepsilon_{\overline{2}} \text{ and } \alpha_i^{D_n} = \varepsilon_{\overline{i+1}} - \varepsilon_{\overline{i}},\ i = 1, ..., n-1 \text{ for the root system } D_n \end{array} \right. . \qquad (4)$$

Then the set of positive roots are

$$\left\{ \begin{array}{l} R_{B_n}^+ = \{\varepsilon_{\overline{i}} - \varepsilon_{\overline{j}}, \varepsilon_{\overline{i}} + \varepsilon_{\overline{j}} \text{ with } 1 \leq j < i \leq n\} \cup \{\varepsilon_{\overline{i}} \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\ R_{B_n}^+ = \{\varepsilon_{\overline{i}} - \varepsilon_{\overline{j}}, \varepsilon_{\overline{i}} + \varepsilon_{\overline{j}} \text{ with } 1 \leq j < i \leq n\} \cup \{2\varepsilon_{\overline{i}} \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ R_{D_n}^+ = \{\varepsilon_{\overline{i}} - \varepsilon_{\overline{j}}, \varepsilon_{\overline{i}} + \varepsilon_{\overline{j}} \text{ with } 1 \leq j < i \leq n\} \text{ for the root system } D_n \end{array} \right. .$$

Denote respectively by $P_{B_n}^+, P_{C_n}^+$ and $P_{D_n}^+$ the sets of dominant weights of so_{2n+1}, sp_{2n} and so_{2n}. Write $\Lambda_0^{B_n}, ..., \Lambda_{n-1}^{B_n}$ for the fundamentals weights of so_{2n+1}, $\Lambda_0^{C_n}, ..., \Lambda_{n-1}^{C_n}$ for the fundamentals weights of sp_{2n} and $\Lambda_0^{D_n}, ..., \Lambda_{n-1}^{D_n}$ for the fundamentals weights of so_{2n+1}.

We have $\Lambda_i^{B_n} = \Lambda_i^{C_n} = \Lambda_i^{D_n} = \varepsilon_{\overline{n}} + \cdots + \varepsilon_{\overline{i+1}}$ for $2 \leq i \leq n-1$, $\Lambda_0^{B_n} = \Lambda_0^{D_n} = \dfrac{1}{2}(\varepsilon_{\overline{n}} + \cdots + \varepsilon_{\overline{2}} + \varepsilon_{\overline{1}})$, $\Lambda_0^{C_n} = \varepsilon_{\overline{n}} + \cdots + \varepsilon_{\overline{2}} + \varepsilon_{\overline{1}}$, $\Lambda_1^{B_n} = \Lambda_1^{C_n} = \varepsilon_{\overline{n}} + \cdots + \varepsilon_{\overline{2}}$ and $\Lambda_1^{D_n} = \dfrac{1}{2}(\varepsilon_{\overline{n}} + \cdots + \varepsilon_{\overline{2}} - \varepsilon_{\overline{1}})$.

Consider $\lambda \in P_{B_n}^+$ and write $\lambda = \sum_{i=0}^{n-1} \widehat{\lambda}_i \Lambda_i^B$ with $\widehat{\lambda}_i \in \mathbb{N}$. Set $\lambda_{\overline{1}} = \dfrac{\widehat{\lambda}_0}{2}$ and $\lambda_{\overline{i}} = \dfrac{\widehat{\lambda}_0}{2} + \widehat{\lambda}_1 + \cdots + \widehat{\lambda}_{i-1}$, $i = 2, ..., n$. The dominant weight λ is characterized by the generalized partition $(\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$ such that $\lambda_{\overline{n}} \geq \cdots \geq \lambda_{\overline{1}}$ and $\lambda_{\overline{i}} \in \dfrac{\mathbb{N}}{2}$, $i = 1, ..., n$. In the sequel we will identify λ and $(\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$ by setting $\lambda = (\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$. Then $\lambda = \lambda_{\overline{1}}\varepsilon_{\overline{1}} + \cdots + \lambda_{\overline{n}}\varepsilon_{\overline{n}}$ that is, the λ_i's are the coordinates of λ on the basis $(\varepsilon_{\overline{n}}, ..., \varepsilon_{\overline{1}})$. The half sum of positive roots verifies $\rho_{B_n} = (n - \dfrac{1}{2}, n - \dfrac{3}{2}, ..., \dfrac{1}{2})$.

Consider $\lambda \in P^+_{C_n}$ and write $\lambda = \sum_{i=0}^{n-1} \widehat{\lambda}_i \Lambda_i^C$ with $\widehat{\lambda}_i \in \mathbb{N}$. The dominant weight λ is characterized by the partition $(\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$ where $\lambda_{\overline{1}} = \widehat{\lambda}_0$ and $\lambda_{\overline{i}} = \widehat{\lambda}_0 + \widehat{\lambda}_1 + \cdots + \widehat{\lambda}_{i-1}$, $i = 2, ..., n$. We set $\lambda = (\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$. Then $\lambda = \lambda_{\overline{1}}\varepsilon_{\overline{1}} + \cdots + \lambda_{\overline{n}}\varepsilon_{\overline{n}}$ and the half sum of positive roots verifies $\rho_{C_n} = (n, n-1, ..., 1)$.

Now consider $\lambda \in P^+_{D_n}$ and write $\lambda = \sum_{i=0}^{n-1} \widehat{\lambda}_i \Lambda_i^D$ with $\widehat{\lambda}_i \in \mathbb{N}$. Set $\lambda_{\overline{1}} = \dfrac{\widehat{\lambda}_0 - \widehat{\lambda}_1}{2}$, $\lambda_{\overline{2}} = \dfrac{\widehat{\lambda}_0 + \widehat{\lambda}_1}{2}$ and $\lambda_{\overline{i}} = \dfrac{\widehat{\lambda}_0 + \widehat{\lambda}_1}{2} + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_{i-1}$, $i = 3, ..., n$. The dominant weight λ is characterized by the generalized partition $(\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$ such that $\lambda_{\overline{n}} \geq \cdots \geq \lambda_{\overline{1}}$, $\lambda_{\overline{i}} \in \dfrac{\mathbb{N}}{2}$ $i = 2, ..., n$ and $\lambda_{\overline{1}} \in \dfrac{\mathbb{Z}}{2}$. Note that we can have $\lambda_{\overline{1}} < 0$. We set $\lambda = (\lambda_{\overline{n}}, ..., \lambda_{\overline{1}})$. Then $\lambda = \lambda_{\overline{1}}\varepsilon_{\overline{1}} + \cdots + \lambda_{\overline{n}}\varepsilon_{\overline{n}}$ and the half sum of positive roots verifies $\rho_{D_n} = (n-1, n-2, ..., 0)$.

For any generalized partition $\lambda = (\lambda_{\overline{n}}, ..., \lambda_{\overline{1}}) \in P_n^+$, we write $\lambda' \in P^+_{n-1}$ for the generalized partition obtained by deleting $\lambda_{\overline{n}}$ in λ. Moreover we set $|\lambda| = \lambda_{\overline{1}} + \lambda_{\overline{2}} + \cdots + \lambda_{\overline{n}}$ if $\lambda_{\overline{1}} \geq 0$, $|\lambda| = -\lambda_{\overline{1}} + \lambda_{\overline{2}} + \cdots + \lambda_{\overline{n}}$ otherwise.
The Weyl group $W_{B_n} = W_{C_n}$ of so_{2n+1} can be regarded as the sub group of the permutation group of $\{\overline{n}, ..., \overline{2}, \overline{1}, 1, 2, ..., n\}$ generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, ..., n-1$ and $s_0 = (1, \overline{1})$ where for $a \neq b$ (a, b) is the simple transposition which switches a and b. We denote by l_B the length function corresponding to the set of generators s_i, $i = 0, ...n-1$.
The Weyl group W_{D_n} of so_{2n} can be regarded as the sub group of the permutation group of $\{\overline{n}, ..., \overline{2}, \overline{1}, 1, 2, ..., n\}$ generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, ..., n-1$ and $s_0' = (1, \overline{2})(2, \overline{1})$. We denote by l_D the length function corresponding to the set of generators s_0' and s_i, $i = 1, ...n-1$.
Note that $W_{D_n} \subset W_{B_n}$ and any $\sigma \in W_{B_n}$ verifies $\sigma(\overline{i}) = \overline{\sigma(i)}$ for $i \in \{1, ..., n\}$. The action of σ on $\beta = (\beta_{\overline{n}}, ..., \beta_{\overline{1}}) \in P_n$ is given by

$$\sigma \cdot (\beta_{\overline{n}}, ..., \beta_{\overline{1}}) = (\beta^{\sigma}_{\overline{n}}, ..., \beta^{\sigma}_{\overline{1}})$$

where $\beta^{\sigma}_{\overline{i}} = \beta_{\sigma(\overline{i})}$ if $\sigma(\overline{i}) \in \{\overline{1}, ..., \overline{n}\}$ and $\beta^{\sigma}_{\overline{i}} = -\beta_{\sigma(i)}$ otherwise.
For any $\beta = (\beta_{\overline{n}}, ..., \beta_{\overline{1}}) \in P_n$ we set $x^{\beta} = x_n^{\beta_{\overline{n}}} \cdots x_1^{\beta_{\overline{1}}}$ where $x_1, ..., x_n$ are fixed indeterminates.
The following lemma is a consequence of Definition 2.1.1.

Proposition 2.2.1 *The Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ is monic of degree*

- $\sum_{i=1}^{n} i(\lambda_{\overline{i}} - \mu_{\overline{i}})$ *for the root system B_n*
- $\sum_{i=1}^{n} i(\lambda_{\overline{i}} - \mu_{\overline{i}}) - \dfrac{1}{2}(|\lambda| - |\mu|)$ *for the root system C_n*
- $\sum_{i=2}^{n} (i-1)(\lambda_{\overline{i}} - \mu_{\overline{i}})$ *for the root system D_n*

Proof. It is similar to that given in Example 4 page 243 of [19] for the degree of Kostka-Foulkes polynomials associated to the root system A_n. ■

Remarks:
(i) : The above proposition suffices to determinate $K_{\lambda,\mu}(q)$ when $\dim V(\lambda)_\mu = 1$. In particular we have $K_{\lambda,\mu}(q) = 1$ for each minuscule representation $V(\lambda)$.
(ii) : If $|\lambda| = |\mu|$ then $K^{B_n}_{\lambda,\mu}(q) = K^{C_n}_{\lambda,\mu}(q) = K^{D_n}_{\lambda,\mu}(q) = K^{A_{n-1}}_{\lambda,\mu}(q)$.
(iii) : Suppose $\lambda, \mu \in P^+_{D_n}$. Set $\lambda^* = (\lambda_{\overline{n}}, ..., \lambda_{\overline{2}}, -\lambda_{\overline{1}})$ and $\mu^* = (\mu_{\overline{n}}, ..., \mu_{\overline{2}}, -\mu_{\overline{1}})$ then

$$K^{D_n}_{\lambda,\mu}(q) = K^{D_n}_{\lambda^*,\mu^*}(q) \tag{5}$$

This is due to the symmetric role played by the simple roots α_0 and $\alpha_{\overline{1}}$ in the root system D_n. Moreover when $\lambda_{\overline{1}} = 0$, $\lambda = \lambda^*$ thus $K^{D_n}_{\lambda,\mu}(q) = K^{D_n}_{\lambda^*,\mu^*}(q) = K^{D_n}_{\lambda^*,\mu}(q) = K^{D_n}_{\lambda,\mu^*}(q)$.
(iv) : Consider $\lambda, \mu \in P_n^+$ such that $\lambda_{\overline{n}} = \mu_{\overline{n}}$. Then $K_{\lambda,\mu}(q) = K_{\lambda',\mu'}(q)$.

2.3 Convention for crystal graphs

In the sequel g is any of the Lie algebras so_{2n+1}, sp_{2n} or so_{2n}. The crystal graphs for the $U_q(g)$-modules are oriented colored graphs with colors $i \in \{0, ..., n-1\}$. An arrow $a \xrightarrow{i} b$ means that $\widetilde{f}_i(a) = b$ and $\widetilde{e}_i(b) = a$ where $\widetilde{e}_i$ and $\widetilde{f}_i$ are the crystal graph operators (for a review of crystal bases and crystal graphs see [7]). A vertex $v^0 \in B$ satisfying $\widetilde{e}_i(v^0) = 0$ for any $i \in \{0, ..., n-1\}$ is called a highest weight vertex. The decomposition of V into its irreducible components is reflected into the decomposition of B into its connected components. Each connected component of B contains a unique highest weight vertex. The crystals graphs of two isomorphic irreducible components are isomorphic as oriented colored graphs. The action of $\widetilde{e}_i$ and $\widetilde{f}_i$ on $B \otimes B' = \{b \otimes b';\ b \in B, b' \in B'\}$ is given by:

$$\widetilde{f}_i(u \otimes v) = \begin{cases} \widetilde{f}_i(u) \otimes v \text{ if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes \widetilde{f}_i(v) \text{ if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases} \tag{6}$$

and

$$\widetilde{e}_i(u \otimes v) = \begin{cases} u \otimes \widetilde{e}_i(v) \text{ if } \varphi_i(u) < \varepsilon_i(v) \\ \widetilde{e}_i(u) \otimes v \text{ if } \varphi_i(u) \geq \varepsilon_i(v) \end{cases} \tag{7}$$

where $\varepsilon_i(u) = \max\{k; \widetilde{e}_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k; \widetilde{f}_i^k(u) \neq 0\}$. The weight of the vertex u is defined by $\mathrm{wt}(u) = \sum_{i=0}^{n-1} (\varphi_i(u) - \varepsilon_i(u))\Lambda_i$.

The following lemma is a straightforward consequence of (6) and (7).

Lemma 2.3.1 *Let $u \otimes v \in B \otimes B'$ $u \otimes v$ is a highest weight vertex of $B \otimes B'$ if and only if for any $i \in \{0, ..., n-1\}$ $\widetilde{e}_i(u) = 0$ (i.e. u is of highest weight) and $\varepsilon_i(v) \leq \varphi_i(u)$.*

The Weyl group W acts on B by:

$$\begin{aligned} s_i(u) &= (\widetilde{f}_i)^{\varphi_i(u)-\varepsilon_i(u)}(u) \text{ if } \varphi_i(u) - \varepsilon_i(u) \geq 0, \\ s_i(u) &= (\widetilde{e}_i)^{\varepsilon_i(u)-\varphi_i(u)}(u) \text{ if } \varphi_i(u) - \varepsilon_i(u) < 0. \end{aligned} \tag{8}$$

We have the equality $\mathrm{wt}(\sigma(u)) = \sigma(\mathrm{wt}(u))$ for any $\sigma \in W$ and $u \in B$. For any $\lambda \in P^+$, we denote by $B(\lambda)$ the crystal graph of $V(\lambda)$.

2.4 Kashiwara-Nakashima's tableaux

Accordingly to (3) the crystal graphs of the vector representations are:

$$B(\Lambda_{n-1}^B) : \overline{n} \overset{n-1}{\to} \overline{n-1} \overset{n-2}{\to} \cdots \to \overline{2} \overset{1}{\to} \overline{1} \overset{0}{\to} 0 \overset{0}{\to} 1 \overset{1}{\to} 2 \cdots \overset{n-2}{\to} n-1 \overset{n-1}{\to} n$$

$$B(\Lambda_{n-1}^C) : \overline{n} \overset{n-1}{\to} \overline{n-1} \overset{n-2}{\to} \cdots \to \overline{2} \overset{1}{\to} \overline{1} \overset{0}{\to} 1 \overset{1}{\to} 2 \cdots \overset{n-2}{\to} n-1 \overset{n-1}{\to} n$$

$$\begin{tikzcd}[column sep=small] & & & & & 1 \arrow[dr, "1"] & & & & & \\ B(\Lambda_{n-1}^D) : \overline{n} \arrow[r, "n-1"] & \overline{n-1} \arrow[r, "n-2"] & \cdots \arrow[r, "3"] & \overline{3} \arrow[r, "2"] & \overline{2} \arrow[ur, "0"] \arrow[dr, "1"'] & & 2 \arrow[r, "2"] & 3 \arrow[r, "3"] & \cdots \arrow[r, "n-2"] & n-1 \arrow[r, "n-1"] & n. \\ & & & & & \overline{1} \arrow[ur, "0"'] & & & & & \end{tikzcd}$$

Kashiwara-Nakashima's combinatorial description of the crystal graphs $B(\lambda)$ is based on a notion of tableaux analogous for each root system B_n, C_n or D_n to semi-standard tableaux.

We define an order on the vertices of the above crystal graphs by setting

$$\mathcal{A}_n^B = \{\overline{n} < \cdots < \overline{1} < 0 < 1 < \cdots < n\}$$

$$\mathcal{A}_n^C = \{\overline{n} < \cdots < \overline{1} < 1 < \cdots < n\} \text{ and}$$

$$\mathcal{A}_n^D = \{\overline{n} < \cdots < \overline{2} < \begin{matrix} 1 \\ \overline{1} \end{matrix} < 2 < \cdots < n\}.$$

Note that $\mathcal{A}_n^D$ is only partially ordered. For any letter x we set $\overline{\overline{x}} = x$. Our convention for labelling the crystal graph of the vector representations are not those used by Kashiwara and Nakashima. To obtain the original description of $B(\lambda)$ from that used in the sequel it suffices to change each letter $k \in \{1, ..., n\}$ into $\overline{n-k+1}$ and each letter $\overline{k} \in \{\overline{1}, ..., \overline{n}\}$ into $n-k+1$. The interest of this change of convention is to yield a natural extension of the above alphabets.

For types B_n, C_n and D_n, we identify the vertices of the crystal graph $G_n^B = \bigoplus_l B(\Lambda_{n-1}^B)^{\otimes l}, G_n^C = \bigoplus_l B(\Lambda_{n-1}^C)^{\otimes l}$ and $G_n^D = \bigoplus_l B(\Lambda_{n-1}^D)^{\otimes l}$ respectively with the words on $\mathcal{A}_n^B, \mathcal{A}_n^C$ and $\mathcal{A}_n^D$. For any $w \in G_n$ we have $\mathrm{wt}(w) = d_{\overline{n}}\varepsilon_{\overline{n}} + d_{\overline{n-1}}\varepsilon_{\overline{n-1}} \cdots + d_{\overline{1}}\varepsilon_{\overline{1}}$ where for all $i = 1, ..., n$ $d_{\overline{i}}$ is the number of letters $\overline{i}$ of w minus its number of letters i.
Consider λ a generalized partition with nonnegative parts. Suppose first that λ is a partition. Write T_λ for the filling of the Young diagram of shape λ whose k-th row contains only letters $\overline{n-k+1}$. Let b_λ be the vertex of $B(\Lambda_{n-1})^{\otimes |\lambda|}$ obtained by column reading T_λ from right to left and top to bottom. Kashiwara and Nakashima realize $B(\lambda)$ as the connected component of the tensor power $B(\Lambda_{n-1})^{\otimes |\lambda|}$ of highest weight vertex b_λ. For each roots system B_n, C_n and D_n, the Kashiwara-Nakashima tableaux of type B_n, C_n, D_n and shape λ are defined as the tableaux whose column readings are the vertices of $B(\lambda)$. We will denote by $\mathrm{w}(T)$ the column reading of the tableau T.
Now suppose that λ belongs to $P_+^{B_n}$ or $P_+^{D_n}$ and its parts are half nonnegative integers. In this case we can write $\lambda = \lambda^\circ + (1/2, ..., 1/2)$ with λ° a partition and $B(\lambda)$ can be realized as the connected component of the crystal graph $\mathfrak{G}_n^0 = B(\Lambda_{n-1})^{\otimes |\lambda^\circ|} \otimes B(\Lambda_0)$ of highest weight vertex $b_\lambda = b_{\lambda^\circ} \otimes b_{\Lambda_0}$ where b_{Λ_0} is the highest weight vertex of $B(\Lambda_0)$ the crystal graph of the spin representation $V(\Lambda_0)$ of the corresponding Lie algebra. The vertices of $B(\Lambda_0)$ are labelled by spin columns which are special column shaped diagrams of width 1/2 and height n. Then the vertices of $B(\lambda)$ can be identified with the column readings of Kashiwara-Nakashima's spin tableaux of types B_n, D_n and shape λ obtained by adding a column shape diagram of width 1/2 to the Young diagram associated to λ°.
Finally suppose that λ belongs to $P_+^{D_n}$ and verifies $\lambda_{\overline{1}} < 0$. The above description of $B(\lambda)$ remain valuable up to the following minor modifications. If the parts of λ are integers the letters $\overline{1}$ must be changed into letters 1 in the above definition of T_λ. Otherwise we set $\lambda = \lambda^\circ + (1/2, ..., 1/2, -1/2)$ where λ° is generalized partition with integer parts. Then $B(\lambda)$ is realized as the connected component of the crystal graph $\mathfrak{G}_n^1 = B(\Lambda_{n-1})^{\otimes |\lambda^\circ|} \otimes B(\Lambda_1^{D_n})$ of highest weight vertex $b_\lambda = b_{\lambda^\circ} \otimes b_{\Lambda_1^{D_n}}$ where $b_{\Lambda_1^{D_n}}$ is the highest weight vertex of $B(\Lambda_1^{D_n})$ the crystal graph of the spin representation $V(\Lambda_1^{D_n})$.

For any generalized partition λ of length n, write $\mathbf{T}^{B_n}(\lambda)$, $\mathbf{T}^{C_n}(\lambda)$ and $\mathbf{T}^{D_n}(\lambda)$ respectively for the sets of Kashiwara-Nakashima's tableaux of shape λ. Set $\mathbf{T}^{B_n} = \bigcup_{\lambda \in P_{B_n}^+} \mathbf{T}^{B_n}(\lambda)$, $\mathbf{T}^{C_n} = \bigcup_{\lambda \in P_{C_n}^+} \mathbf{T}^{C_n}(\lambda)$ and $\mathbf{T}^{D_n} = \bigcup_{\lambda \in P_{D_n}^+} \mathbf{T}^{D_n}(\lambda)$. In the sequel we only summarize the combinatorial description of the partition shaped tableaux that is, tableaux of $\mathbf{T}^n(\lambda)$ where the parts of λ are integers (with eventually $\lambda_{\overline{1}} < 0$ for the root system D_n). We refer the reader to [1], [9], [13] and [14] for the complete description of $\mathbf{T}^n(\lambda)$ which necessitates a large amount of combinatorial definitions especially when the parts of λ are half integers.
So consider λ a generalized partition with integer parts. Suppose first that $\lambda_{\overline{n}} = 1$. Then the tableaux of $\mathbf{T}^n(\lambda)$ are called the n-admissible columns. The n-admissible columns of types B_n, C_n and D_n are in particular columns of types B_n, C_n and D_n that is have the form

$$C = \begin{array}{|c|}\hline C_- \\ \hline C_0 \\ \hline C_+ \\ \hline \end{array}, C = \begin{array}{|c|}\hline C_- \\ \hline C_+ \\ \hline \end{array} \text{ and } C = \begin{array}{|c|}\hline D_- \\ \hline D \\ \hline D_+ \\ \hline \end{array} \tag{9}$$

where C_-, C_+, C_0, D_-, D_+ and D are column shaped Young diagrams such that

$$\begin{cases} C_- \text{ is filled by strictly increasing barred letters from top to bottom} \\ C_+ \text{ is filled by strictly increasing unbarred letters from top to bottom} \\ C_0 \text{ is filled by letters } 0 \\ D_- \text{ is filled by strictly increasing letters } \leq \overline{2} \text{ from top to bottom} \\ D_+ \text{ is filled by strictly increasing letters } \geq 2 \text{ from top to bottom} \\ D \text{ is filled by letters } \overline{1} \text{ or } 1 \text{ with differents letters in two adjacent boxes} \end{cases}.$$

Note that all the columns are not n-admissible even if their letters a satisfy $\overline{n} \leq a \leq n$. More precisely a column C of (9) is n-admissible if and only if it can be duplicated following a simple algorithm described in [14] into a pair (lC, rC) of columns without pair of opposite letters $(x, \overline{x})$ (the letter 0 is counted as the pair $(0, \overline{0})$) and containing only letters a such that $\overline{n} \leq a \leq n$.

Example 2.4.1 *For the column* $C = \begin{array}{|c|}\hline \overline{3} \\ \hline \overline{1} \\ \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$ *of type B we have* $lC = \begin{array}{|c|}\hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$ *and* $rC = \begin{array}{|c|}\hline \overline{3} \\ \hline \overline{1} \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$. *Hence C is 5-admissible but not n-admissible for $n \leq 4$.*

Now for a general λ with integer parts, a tableau $T \in \mathbf{T}^n(\lambda)$ can be regarded as a filling of the Young diagram of shape λ if $\lambda_{\overline{1}} \geq 0$ (of shape λ^* otherwise) such that

- $T = C_1 \cdots C_r$ where the columns C_i of T are n-admissible,
- for any $i \in \{1, ...r-1\}$ the columns of the tableau $r(C_i)l(C_{i+1})$ weakly increase from left to right and do not contain special configurations (detailed in [9] and [14]) when T is of type D_n.

Remark: We have $\mathbf{T}^n(\lambda) \subset \mathbf{T}^{n+1}(\lambda^\#)$ where $\lambda^\# = (\lambda_{\overline{n}}, ..., \lambda_{\overline{1}}, 0)$ since the n-admissible columns are also $(n+1)$-admissible and the duplication process of a column does not depend on n. To simplify the notation we will write in the sequel $\mathbf{T}^{n+1}(\lambda)$ instead of $\mathbf{T}^{n+1}(\lambda^\#)$ for any $\lambda \in P_n^+$.

2.5 Insertion schemes and plactic monoids

There exist insertion schemes related to each classical root system [1], [13] and [14] analogous for Kashiwara-Nakashima's tableaux to the well known bumping algorithm on semi-standard tableaux.
Denote by $\sim_n^B, \sim_n^C$ and $\sim_n^D$ the equivalence relations defining on the vertices of G_n^B, G_n^C and G_n^D by $w_1 \sim_n w_2$ if and only if w_1 and w_2 belong to the same connected component of G_n. For any word w, the insertions schemes permit to compute the unique tableau $P_n(w)$ such that $w \sim_n \mathrm{w}(P_n(w))$. In fact $\sim_n^B, \sim_n^C$ and $\sim_n^D$ are congruencies $\equiv_n^B, \equiv_n^C$ and $\equiv_n^D$ [1] [13] [14] [16] obtained respectively as the quotient of the free monoids of words on $\mathcal{A}_n^C, \mathcal{A}_n^B$ and $\mathcal{A}_n^D$ by two kinds of relations.
The first is constituted by relations of length 3 analogous to Knuth relations defining Lascoux-Schützenberger's plactic monoid. In fact these relations are precisely those which are needed to describe the insertion $x \to C$ of a letter x in a n-admissible column $C = \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}$ such that $\begin{array}{|c|}\hline a \\ \hline b \\ \hline x \\ \hline \end{array}$ is not a column. This can be written

$$x \to \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} = \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{a} \\ \hline \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{b} \\ \hline \end{array} = \begin{array}{cc} \hline \multicolumn{1}{|c|}{a'} & \multicolumn{1}{c|}{x'} \\ \hline \multicolumn{1}{|c|}{b'} & \\ \cline{1-1} \end{array} \tag{10}$$

and contrary to the insertion scheme for the semi-standard tableaux the sets $\{a', b', x'\}$ and $\{a, b, c\}$ are not necessarily equal (i.e. the relations are not homogeneous in general).
Next we have the contraction relations which do not preserve the length of the words. These relations are precisely those which are needed to describe the insertion $x \to C$ of a letter x such that $\overline{n} \leq x \leq n$ in a n-admissible column C such that $\begin{array}{|c|}\hline C \\ \hline x \\ \hline \end{array}$ (obtained by adding the letter x on bottom of C) is a column which is not n-admissible. In this case $\begin{array}{|c|}\hline C \\ \hline x \\ \hline \end{array}$ is necessarily $(n+1)$-admissible and have to be contracted to give a n-admissible column. We obtain $x \to C = \widetilde{C}$ with $\widetilde{C}$ a n-admissible column of height $h(C)$ or $h(C) - 1$.
The insertion of the letter x in a n-admissible column C of arbitrary height such that $\begin{array}{|c|}\hline C \\ \hline x \\ \hline \end{array}$ is not a column can

then be pictured by

$$x \rightarrow \begin{array}{|c|}\hline a_1\\\hline \cdot\\\hline a_{k-2}\\\hline a_{k-1}\\\hline a_k\\\hline\end{array} = \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a_1}\\\cline{2-2} & \multicolumn{1}{|c|}{\cdot}\\\cline{2-2} & \multicolumn{1}{|c|}{a_{k-2}}\\\cline{2-2} & \multicolumn{1}{|c|}{a_{k-1}}\\\hline \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{a_k}\\\hline\end{array} = \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a_1}\\\cline{2-2} & \multicolumn{1}{|c|}{\cdot}\\\cline{2-2} & \multicolumn{1}{|c|}{a_{k-2}}\\\hline \multicolumn{1}{|c|}{\delta_{k-1}} & \multicolumn{1}{c|}{y}\\\hline \multicolumn{1}{|c|}{d_k} & \\\cline{1-1}\end{array} = \cdots = \begin{array}{cc}\hline \multicolumn{1}{|c|}{d_1} & \multicolumn{1}{c|}{z}\\\hline \multicolumn{1}{|c|}{\cdot} & \\\cline{1-1} \multicolumn{1}{|c|}{\cdot} & \\\cline{1-1} \multicolumn{1}{|c|}{d_{k-1}} & \\\cline{1-1} \multicolumn{1}{|c|}{d_k} & \\\cline{1-1}\end{array}$$

that is, one elementary transformation (10) is applied to each step. One proves that $x \rightarrow C$ is then a tableau of $\mathbf{T}^{(n)}$ with two columns respectively of height $h(C)$ and 1.

Now we can define the insertion $x \rightarrow T$ of the letter x such that $\overline{n} \leq x \leq n$ in the tableau $T \in \mathbf{T}^n(\lambda)$. Set $T = C_1 \cdots C_r$ where C_i, $i = 1, ..., r$ are the n-admissible columns of T.

1. When $\begin{array}{|c|}\hline C_1\\\hline x\\\hline\end{array}$ is not a column, write $x \rightarrow C = \begin{array}{|c|c|}\hline C_1' & y\\\hline\end{array}$ where C_1' is an admissible column of height $h(C_1)$ and y a letter. Then $x \rightarrow T = C_1'(y \rightarrow C_2 \cdots C_r)$ that is, $x \rightarrow T$ is the juxtaposition of C_1' with the tableau $\widehat{T}$ obtained by inserting y in the tableau $C_2 \cdots C_r$.

2. When $\begin{array}{|c|}\hline C_1\\\hline x\\\hline\end{array}$ is a n-admissible column, $x \rightarrow T$ is the tableau obtained by adding a box containing x on bottom of C_1.

3. When $\begin{array}{|c|}\hline C_1\\\hline x\\\hline\end{array}$ is a column which is not n-admissible, write $x \rightarrow C = \widetilde{C}$ and set $\mathrm{w}(\widetilde{C}) = y_1 \cdots y_s$ where the y_i's are letters. Then $x \rightarrow T = y_s \rightarrow (y_{s-1} \rightarrow (\cdots y_1 \rightarrow \widehat{T}))$ that is $x \rightarrow T$ is obtained by inserting successively the letters of $\widetilde{C}$ into the tableau $\widehat{T} = C_2 \cdots C_r$. Note that there is no new contraction during this s insertions.

Remarks:
(i): The P_n-symbol defined above can be computed recursively by setting $P_n(w) = \begin{array}{|c|}\hline w\\\hline\end{array}$ if w is a letter and $P_n(w) = x \rightarrow P_n(u)$ where $w = ux$ with u a word and x a letter otherwise.
(ii): Consider $T \in \mathbf{T}^n(\lambda) \subset \mathbf{T}^{n+1}(\lambda)$ and a letter x such that $\overline{n} \leq x \leq n$. The tableau obtained by inserting x in T may depend wether T is regarded as a tableau of $\mathbf{T}^n(\lambda)$ or as a tableau of $\mathbf{T}^{n+1}(\lambda)$. Indeed if $\begin{array}{|c|}\hline C_1\\\hline x\\\hline\end{array}$ is not n-admissible then it is necessarily $(n+1)$-admissible since C_1 is n-admissible. Hence there is no contraction during the insertion $x \rightarrow T$ when it is regarded as a tableau of $\mathbf{T}^{n+1}(\lambda)$.
(iii): Consider $w \in \mathcal{A}_n$, from (ii) we deduce that there exists an integer $m \geq n$ minimal such that $P_m(w)$ can be computed without using contraction relation. Then for any $k \geq m$, $P_k(w) = P_m(w)$.
(iv): Similarly to the bumping algorithm for semi-standard tableaux, the insertion algorithms described above are reversible.
(v): There also exit insertion algorithms for the spin tableaux of types B_n and D_n [14]. To make the paper more readable we only establish the combinatorial results contained in the sequel for the partition shaped tableaux. Nevertheless note that they can be extended to take also into account the spin tableaux associated to the root systems B_n and D_n.

Lemma 2.5.1 *Consider $\lambda, \mu \in P_n^+$. Let $T \in \mathbf{T}^n(\lambda)$. If λ and μ have integer parts, then there exists a unique pair (R, T') such that*

$$\mathrm{w}(T) \equiv_n \mathrm{w}(R) \otimes \mathrm{w}(T')$$

where $R \in \mathbf{T}^n(\lambda_{\overline{n}} \Lambda_{n-1})$ is a row tableau of length $\lambda_{\overline{n}}$ and $T' \in \mathbf{T}^{n-1}(\lambda')$ with $\lambda' = (\lambda_{\overline{n-1}}, ..., \lambda_{\overline{1}})$.

Proof. When $\lambda_{\overline{1}} \geq 0$ we have

$$b_\lambda \equiv_n (\overline{n})^{\otimes \lambda_{\overline{n}}} \otimes \left((\overline{1})^{\otimes \widehat{\lambda}'_{\overline{1}}} \otimes (\overline{1}\ \overline{2})^{\otimes \widehat{\lambda}'_{\overline{2}}} \otimes \cdots \otimes (\overline{1}\ \overline{2} \cdots \overline{n-1})^{\otimes \widehat{\lambda}'_{\overline{n-1}}} \right) \equiv_n b_{\lambda_{\overline{n}} \Lambda_{n-1}} \otimes b_{\lambda'}$$

with the notation used in 2.2. Indeed the plactic relations on words containing only barred letters coincide with Knuth relations. This implies the existence of the required pair (R, T'). Now if $\mathrm{w}(T) \equiv_n \mathrm{w}(R) \otimes \mathrm{w}(T')$, we deduce from 2.3.1 that the highest weight vertex of the connected component of G_n containing $\mathrm{w}(R) \otimes \mathrm{w}(T')$ is necessarily $b_{\lambda_{\overline{n}}\Lambda_{n-1}} \otimes b_{\lambda'}$. Thus the pair (R, T') is unique. ■

Remark: The pair (R, T') can be explicitly computed by using the reverse insertion schemes.

3 Morris type recurrence formulas for the orthogonal root systems

In this section we introduce recurrence formulas for computing Kostka-Foulkes polynomials analogous for types B_n and D_n to Morris recurrence formula. They allow to explain the Kostka-Foulkes polynomials for types B_n and D_n respectively as combinations of Kostka-Foulkes polynomials for types B_{n-1} and D_{n-1}. We essentially proceed as we have done in [15] for the root system C_n. So we only sketch the arguments except for Theorems 3.2.1 and 3.2.2 for which the proofs necessitate refinements of the proof of Theorem 3.2.1 of [15].

We classically realize so_{2n-1}, sp_{2n-2} and so_{2n-2} respectively as the sub-algebras of so_{2n+1}, sp_{2n} and so_{2n} generated by the Chevalley operators e_i, f_i and t_i, $i = 0, \ldots n-2$. The weight lattice P_{n-1} of these algebras of rank $n-1$ is the $\mathbb{Z}$-lattice generated by the $\varepsilon_{\overline{i}}$, $i = 1, \ldots, n-1$ and $P_{n-1}^+ = P_n^+ \cap P_{n-1}$ is the set of dominant weights. The Weyl group W_{n-1} is the sub-group of W_n generated by the s_i, $i = 0, \ldots n-2$ and we have $R_{n-1}^+ = R_n \cap P_{n-1}^+$. Given any positive integer r, set $B^{B_n}(r) = B(r\Lambda_{n-1}^{B_n}), B^{C_n}(r) = B(r\Lambda_{n-1}^{C_n})$, and $B^{D_n}(r) = B(r\Lambda_{n-1}^{D_n})$. To obtain our recurrence formulas we need to describe the decomposition $B(\gamma) \otimes B(r)$ with $\gamma \in P_n^+$ and $r \geq 0$ an integer into its irreducible components. This is analogous for types B_n and D_n to Pieri rule.

3.1 Pieri rule for types B_n and D_n

It follows from [9] that the vertices of $B^{B_n}(r)$, $B^{C_n}(r)$ and $B^{D_n}(r)$ can be respectively identified to the words

$$L = (n)^{k_n} \cdots (2)^{k_2}(1)^{k_1}(\overline{1})^{k_{\overline{1}}}(\overline{2})^{k_{\overline{2}}} \cdots (\overline{n})^{k_{\overline{n}}},\ L = (n)^{k_n} \cdots (2)^{k_2}(1)^{k_1}(0)(\overline{1})^{k_{\overline{1}}}(\overline{2})^{k_{\overline{2}}} \cdots (\overline{n})^{k_{\overline{n}}} \tag{11}$$

$$L = (n)^{k_n} \cdots (2)^{k_2}(1)^{k_1}(\overline{1})^{k_{\overline{1}}}(\overline{2})^{k_{\overline{2}}} \cdots (\overline{n})^{k_{\overline{n}}} \tag{12}$$

and

$$L = (n)^{k_n} \cdots (2)^{k_2}(\overline{1})^{k_{\overline{1}}}(\overline{2})^{k_{\overline{2}}} \cdots (\overline{n})^{k_{\overline{n}}},\ L = (n)^{k_n} \cdots (2)^{k_2}(1)^{k_1}(\overline{2})^{k_{\overline{2}}} \cdots (\overline{n})^{k_{\overline{n}}} \tag{13}$$

of length r where $k_{\overline{i}}, k_i$ are positive integers, $(x)^k$ means that the letter x is repeated k times in L. Note that there can be only one letter 0 in the vertices of $B^{B_n}(r)$ and the letters $\overline{1}$ and 1 can not appear simultaneously in the vertices of $B^{D_n}(r)$.

Let $\gamma = (\gamma_{\overline{n}}, \ldots, \gamma_{\overline{1}}) \in P_n^+$. When $\gamma \in P_{B_n}^+$ set $B(\gamma) \otimes B^{B_n}(r) = \bigcup_{\lambda \in P_{B_n}^+} B(\lambda)^{b_{\gamma,r}^{\lambda}}$ that is $b_{\gamma,r}^{\lambda}$ is the multiplicity of $V(\lambda)$ in $V(\gamma) \otimes V(r\Lambda_{n-1}^{B})$. Similarly set $B(\gamma) \otimes B^{C_n}(r) = \bigcup_{\lambda \in P_{C_n}^+} B(\lambda)^{c_{\gamma,r}^{\lambda}}$ and $B(\gamma) \otimes B^{D_n}(r) = \bigcup_{\lambda \in P_{D_n}^+} B(\lambda)^{d_{\gamma,r}^{\lambda}}$ when γ belongs respectively to $P_{C_n}^+$ and $P_{D_n}^+$.

Write b_γ for the highest weight vertex of $B(\gamma)$. The two following lemmas and their corollaries are consequences of Lemma 2.3.1.

Lemma 3.1.1 *$b_\gamma \otimes L$ is a highest weight vertex of $B(\gamma) \otimes B^{B_n}(r)$ if and only if the following conditions holds:*
(i) : $\gamma_{\overline{1}} - k_1 \geq 0$ *if* $k_0 = 0$, $\gamma_{\overline{1}} - k_1 > 0$ *otherwise*
(ii) : $\gamma_{\overline{i+1}} - k_{i+1} \geq \gamma_{\overline{i}}$ *for* $i = 1, \ldots, n-1$
(iii) : $\gamma_{\overline{i}} - k_i + k_{\overline{i}} \leq \gamma_{\overline{i+1}} - k_{i+1}$ *for* $i = 1, \ldots, n-1$

Corollary 3.1.2 *The multiplicity $b_{\gamma,r}^{\lambda}$ is the number of vertices $L \in B^{B_n}(r)$ such that $k_{\overline{i}} - k_i = \lambda_{\overline{i}} - \gamma_{\overline{i}}$ for $i = 1, \ldots., n$ and*
(i) : $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - k_{\overline{i+1}}$ *for* $i = 1, \ldots, n-1$,
(ii) : $\lambda_{\overline{i+1}} - k_{\overline{i+1}} \geq \lambda_{\overline{i}} + k_i - k_{\overline{i}}$ *for* $i = 1, \ldots, n-1$,
(iii) : $\lambda_{\overline{1}} - k_{\overline{1}} \geq 0$ *if* $k_0 = 0$ *(i.e.* $k_{\overline{1}} + \cdots + k_{\overline{n}} + k_1 + \cdots + k_n = r$*) and* $\lambda_{\overline{1}} - k_{\overline{1}} > 0$ *otherwise (i.e.* $k_{\overline{1}} + \cdots + k_{\overline{n}} + k_1 + \cdots + k_n = r - 1$*).*

Lemma 3.1.3 *$b_\gamma \otimes L$ is a highest weight vertex of $B(\gamma)\otimes B^{D_n}(r)$ if and only if the following conditions holds:*
(i) : $\gamma_{\overline{2}} - k_2 \geq \gamma_{\overline{1}}$ *if* $\gamma_{\overline{1}} \geq 0$ *and* $\gamma_{\overline{2}} - k_2 \geq -\gamma_{\overline{1}}$ *otherwise,*
(ii) : $\gamma_{\overline{i+1}} - k_{i+1} \geq \gamma_{\overline{i}}$ *for* $i = 2, ..., n-1$,
(iii) :$\gamma_{\overline{1}} + k_{\overline{1}} \leq \gamma_{\overline{2}} - k_2$ *if* $k_1 = 0$ *and* $-\gamma_{\overline{1}} + k_1 \leq \gamma_{\overline{2}} - k_2$ *otherwise,*
(iii) : $\gamma_{\overline{i}} - k_i + k_{\overline{i}} \leq \gamma_{\overline{i+1}} - k_{i+1}$ *for* $i = 2, ..., n-1$

Corollary 3.1.4 *The multiplicity $d^{\lambda}_{\gamma,r}$ is the number of vertices $L \in \otimes B^{D_n}(r)$ such that $k_{\overline{i}} - k_i = \lambda_{\overline{i}} - \gamma_{\overline{i}}$ for $i = 1, ..., n$, and*
(i) : $\lambda_{\overline{1}} \leq \lambda_{\overline{2}} - k_{\overline{2}}$ *if* $k_1 = 0$ *and* $-\lambda_{\overline{1}} \leq \lambda_{\overline{2}} - k_{\overline{2}}$ *otherwise*
(ii) : $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - k_{\overline{i+1}}$ *for* $i = 2, ..., n-1$,
(iii) : $\lambda_{\overline{i+1}} - k_{\overline{i+1}} \geq \lambda_{\overline{i}} + k_i - k_{\overline{i}}$ *for* $i = 2, ..., n-1$,
(iv) : $\left\{\begin{array}{l}(a) : \lambda_{\overline{2}} - k_{\overline{2}} \geq \lambda_{\overline{1}} - k_{\overline{1}} \text{ if } k_1 = 0 \text{ and } \gamma_{\overline{1}} \geq 0\\ (b) : \lambda_{\overline{2}} - k_{\overline{2}} \geq \lambda_{\overline{1}} + k_1 \text{ if } k_{\overline{1}} = 0 \text{ and } \gamma_{\overline{1}} \geq 0\end{array}\right\}, \left\{\begin{array}{l}(c) : \lambda_{\overline{2}} - k_{\overline{2}} \geq -\lambda_{\overline{1}} - k_1 \text{ if } k_{\overline{1}} = 0 \text{ and } \gamma_{\overline{1}} < 0\\ (d) : \lambda_{\overline{2}} - k_{\overline{2}} \geq -\lambda_{\overline{1}} + k_{\overline{1}} \text{ if } k_1 = 0 \text{ and } \gamma_{\overline{1}} < 0\end{array}\right\}$.

Remarks:
(i) : In the above corollaries, $b^{\lambda}_{\gamma,r}$ and $d^{\lambda}_{\gamma,r}$ are the number of ways of starting with γ, removing a horizontal strip to obtain a partition ν (corresponding to the unbarred letters of L) and then adding a horizontal strip (corresponding to the barred letters of L) to obtain λ.
(ii) : $B(\gamma) \otimes B((r)_n)$ is not multiplicity free in general.
(iii) : Consider $\gamma = (\gamma_{\overline{n}}, ..., \gamma_{\overline{1}}) \in P_{B_n}$ (resp. P_{D_n}) such that $\lambda = (\lambda_{\overline{n}}, ..., \lambda_{\overline{1}}) \in P_{B_n}$ (resp. P_{D_n}) defined by $\lambda_{\overline{i}} = \gamma_{\overline{i}} + k_{\overline{i}} - k_{\overline{i}}$, $i = 1, ..., n$ verifies conditions (i), (ii) and (iii) of Corollary 3.1.2 (resp. 3.1.4). Then $\gamma \in P^+_{B_n}$ (resp. $P^+_{D_n}$) that is γ is a generalized partition.

3.2 Recurrence formulas

Consider $\gamma \in P^+_n$ and r a positive integer. We set

$$(\gamma \otimes r)_{B_n} = \{\lambda \in P^+_{B_n},\ b^{\lambda}_{\gamma,r} \neq 0\}, (\gamma \otimes r)_{C_n} = \{\lambda \in P^+_{C_n},\ c^{\lambda}_{\gamma,r} \neq 0\}$$
$$\text{and } (\gamma \otimes r)_{D_n} = \{\lambda \in P^+_{D_n},\ d^{\lambda}_{\gamma,r} \neq 0\}.$$

For the root system C_n and $\mu = (\mu_{\overline{n}}, ..., \mu_{\overline{1}})$, we have established in [15] the following analogue of Morris recurrence formula:

$$Q^{C_n}_{\mu} = \sum_{\gamma \in P^+_{C_{n-1}}} \sum_{R=0}^{+\infty} \sum_{r+2m=R} q^{m+r} \sum_{\lambda \in (\gamma \otimes r)_{C_{n-1}}} c^{\lambda}_{\gamma,r} K^{C_{n-1}}_{\lambda,\mu'}(q) s_{(\mu_{\overline{n}}+R,\gamma)}$$

where $\mu' = (\mu_{\overline{n-1}}, ..., \mu_{\overline{1}}) \in P^+_{C_{n-1}}$.

Theorem 3.2.1 *Let $\mu \in P^+_{B_n}$. Then*

$$Q^{B_n}_{\mu} = \sum_{\gamma \in P^+_{B_{n-1}}} \sum_{R=0}^{+\infty} \sum_{r+2m=R} q^{R} \sum_{\lambda \in (\gamma \otimes r)_{B_{n-1}}} b^{\lambda}_{\gamma,r} K^{B_{n-1}}_{\lambda,\mu'}(q) s_{(\mu_{\overline{n}}+R,\gamma)}. \tag{14}$$

Proof. From $Q_\mu = \left(\prod_{\alpha \in R^+_{B_n}} \dfrac{1}{1-qR_\alpha}\right) s_\mu$ and Proposition 3.5 of [21] we can write

$$Q_\mu = \left(\prod_{\substack{\alpha \in R^+_{B_n}\\ \alpha \notin R^+_{B_{n-1}}}} \frac{1}{1-qR_\alpha}\right)\left[\left(\prod_{\alpha \in R^+_{B_{n-1}}} \frac{1}{1-qR_\alpha}\right) s_\mu\right].$$

Then by applying Theorem 2.1.2, we obtain

$$Q_\mu = \left(\prod_{\substack{\alpha \in R^+_{B_n} \\ \alpha \notin R^+_{B_{n-1}}}} \frac{1}{1 - qR_\alpha} \right) \left(\sum_{\lambda \in P^+_{B_{n-1}}} K^{B_{n-1}}_{\lambda,\mu'}(q) s_{(\mu_{\overline{n}},\lambda)} \right). \tag{15}$$

Set $R_{\overline{i}} = R_{\varepsilon_{\overline{n}} - \varepsilon_{\overline{i}}}$ for $i = 1, ..., n-1$ $R_n = R_{\varepsilon_{\overline{n}}}$ and $R_i = R_{\varepsilon_{\overline{n}} + \varepsilon_{\overline{i}}}$ for $i = 1, ..., n$. Recall that for any $\beta \in P_{B_{n-1}}$, $R_{\overline{i}}(\beta) = \beta + \varepsilon_{\overline{n}} - \varepsilon_{\overline{i}}$ and $R_i(\beta) = \beta + \varepsilon_{\overline{n}} + \varepsilon_{\overline{i}}$. Then (15) implies

$$Q_\mu = \sum_{\lambda \in P^+_{B_{n-1}}} K^{B_{n-1}}_{\lambda,\mu'}(q) \times$$
$$\left(\sum_{r=0}^{+\infty} \sum_{b=0}^{+\infty} \sum_{k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r} q^{r+b} (R_n)^b (R_1)^{k_1} (R_{\overline{1}})^{k_{\overline{1}}} \cdots (R_{n-1})^{k_{n-1}} (R_{\overline{n-1}})^{k_{\overline{n-1}}} s_{(\mu_{\overline{n}},\lambda)} \right).$$

$$Q_\mu = \sum_{r=0}^{+\infty} \sum_{b=0}^{+\infty} q^{r+b} \sum_{\lambda \in P^+_{B_{n-1}}} K^{B_{n-1}}_{\lambda,\mu'}(q) \sum_{k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r} s_{(\mu_{\overline{n}} + r + b, \lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, \cdots, \lambda_{\overline{1}} + k_1 - k_{\overline{1}})} =$$
$$\sum_{R=0}^{+\infty} \sum_{r=0}^{R} q^R \sum_{\lambda \in P^+_{B_{n-1}}} K^{B_{n-1}}_{\lambda,\mu'}(q) \sum_{k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r} s_{(\mu_{\overline{n}} + R, \lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, \cdots, \lambda_{\overline{1}} + k_1 - k_{\overline{1}})}$$

by setting $R = r + b$. Now fix $\lambda, R > 0$ and $0 < r \leq R$ and write

$$S_1 = \sum_{k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r} s_{(\mu_{\overline{n}} + R, \lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, \cdots, \lambda_{\overline{1}} + k_1 - k_{\overline{1}})},$$
$$S_2 = \sum_{k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r-1} s_{(\mu_{\overline{n}} + R, \lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, \cdots, \lambda_{\overline{1}} + k_1 - k_{\overline{1}})},$$

$S_{R,r} = S_1 + S_2$ and $\gamma = (\lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, ..., \lambda_{\overline{1}} + k_1 - k_{\overline{1}})$.
(a) : Consider γ appearing in S_1 or S_2 and suppose that there exists $i \in \{1, ..., n-2\}$ such that $\lambda_{\overline{i}} > \lambda_{\overline{i+1}} - k_{\overline{i+1}}$. Set $\widetilde{\gamma} = s_i \circ \gamma$ that is

$$\widetilde{\gamma} = s_i(\gamma_{\overline{n-1}} + n - 3/2, ..., \gamma_{\overline{i+1}} + n - i + 1/2, \gamma_{\overline{i}} + n - i, ..., \gamma_{\overline{1}} + 1/2) - (n - 3/2, ..., 1/2).$$

Then $\gamma_{\overline{s}} = \widetilde{\gamma}_{\overline{s}}$ for $s \neq i+1, i$, $\widetilde{\gamma}_{\overline{i+1}} = \gamma_{\overline{i}} - 1$ and $\widetilde{\gamma}_{\overline{i}} =, \gamma_{\overline{i+1}} + 1$ that is

$$\begin{cases} \widetilde{\gamma}_{\overline{i+1}} = \lambda_{\overline{i}} + k_i - k_{\overline{i}} - 1 \\ \widetilde{\gamma}_{\overline{i}} = \lambda_{\overline{i+1}} + k_{i+1} - k_{\overline{i+1}} + 1 \end{cases}.$$

Write $\widetilde{k}_{i+1} = k_i$, $\widetilde{k}_i = k_{i+1}$, $\widetilde{k}_{\overline{i+1}} = \lambda_{\overline{i+1}} - \lambda_{\overline{i}} + k_{\overline{i}} + 1$ and $\widetilde{k}_{\overline{i}} = \lambda_{\overline{i}} - \lambda_{\overline{i+1}} + k_{\overline{i+1}} - 1$. To make our notation homogeneous set $\widetilde{k}_t = k_t$ for any $t \neq i, i+1, \overline{i}, \overline{i+1}$. Then $\lambda_{\overline{i}} > \lambda_{\overline{i+1}} - \widetilde{k}_{\overline{i+1}}$. We have $\widetilde{k}_{\overline{i+1}} \geq 0$ and $\widetilde{k}_{\overline{i}} = \lambda_{\overline{i}} - \lambda_{\overline{i+1}} + k_{\overline{i+1}} - 1 \geq 0$ since $\lambda_{\overline{i}} > \lambda_{\overline{i+1}} - k_{\overline{i+1}}$. Moreover $\widetilde{k}_{\overline{1}} + \cdots + \widetilde{k}_{\overline{n-1}} + \widetilde{k}_1 + \cdots + \widetilde{k}_{n-1} = k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1}$ and for any $s \in \{1, ..., n-2\}$

$$\widetilde{\gamma}_{\overline{s}} = \lambda_{\overline{s}} + \widetilde{k}_s - \widetilde{k}_{\overline{s}}.$$

(b) : Consider γ appearing in S_1 and suppose that $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - k_{\overline{i+1}}$ for all $i = 1, ..., n-2$ and $\lambda_{\overline{1}} - k_{\overline{1}} < 0$. Set $\widetilde{\gamma} = s_0 \circ \gamma$. Then $\gamma_{\overline{s}} = \widetilde{\gamma}_{\overline{s}}$ for $s \neq 1$ and $\widetilde{\gamma}_{\overline{1}} = -\lambda_{\overline{1}} - k_1 + k_{\overline{1}} - 1$. Write $\widetilde{k}_i = k_i$, $\widetilde{k}_{\overline{i}} = k_{\overline{i}}$ for all $i = 2, ..., n-1$ and set $\widetilde{k}_1 = k_{\overline{1}} - \lambda_{\overline{1}} - 1$, $\widetilde{k}_{\overline{1}} = k_1 + \lambda_{\overline{1}}$. We have $\widetilde{k}_1 \geq 0$, $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - \widetilde{k}_{\overline{i+1}}$ for all $i = 1, ..., n-2$ and $\lambda_{\overline{1}} - \widetilde{k}_{\overline{1}} \leq 0$. Moreover $\widetilde{k}_{\overline{1}} + \cdots + \widetilde{k}_{\overline{n-1}} + \widetilde{k}_1 + \cdots + \widetilde{k}_{n-1} = r - 1$ (thus γ appears in S_2) and $\widetilde{\gamma}_{\overline{1}} = \lambda_{\overline{1}} + \widetilde{k}_1 - \widetilde{k}_{\overline{1}}$.

(c) : Consider γ appearing in S_2 and suppose that $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - k_{\overline{i+1}}$ for all $i = 1, ..., n-2$ and $\lambda_{\overline{1}} - k_{\overline{1}} \leq 0$. Set $\widetilde{\gamma} = s_0 \circ \gamma$. Then $\gamma_{\overline{s}} = \widetilde{\gamma}_{\overline{s}}$ for $s \neq 1$ and $\widetilde{\gamma}_{\overline{1}} = -\lambda_{\overline{1}} - k_1 + k_{\overline{1}} - 1$. Write $\widetilde{k}_i = k_i$, $\widetilde{k}_{\overline{i}} = k_{\overline{i}}$ for all $i = 2, ..., n-1$ and set $\widetilde{k}_1 = k_{\overline{1}} - \lambda_{\overline{1}}$, $\widetilde{k}_{\overline{1}} = k_1 + \lambda_{\overline{1}} + 1$. We have $\widetilde{k}_1 \geq 0$, $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - \widetilde{k}_{\overline{i+1}}$ for all $i = 1, ..., n-2$ and $\lambda_{\overline{1}} - \widetilde{k}_{\overline{1}} < 0$. Moreover $\widetilde{k}_{\overline{1}} + \cdots + \widetilde{k}_{\overline{n-1}} + \widetilde{k}_1 + \cdots + \widetilde{k}_{n-1} = r$ (thus γ appears in S_1) and $\widetilde{\gamma}_{\overline{1}} = \lambda_{\overline{1}} + \widetilde{k}_1 - \widetilde{k}_{\overline{1}}$.
(d) : Now consider γ appearing in S_1 or S_2 and suppose that $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - k_{\overline{s+1}}$ for any $s \in \{1, ..., n-2\}$, $\lambda_{\overline{1}} - k_{\overline{1}} \geq 0$ (resp. $\lambda_{\overline{1}} - k_{\overline{1}} > 0$) if γ appears in S_1 (resp. in S_2) and there exists $i \in \{1, ..., n-2\}$ such that $\lambda_{\overline{i+1}} - k_{\overline{i+1}} < \lambda_{\overline{i}} + k_i - k_{\overline{i}}$. Define $\widetilde{\gamma} = s_i \circ \gamma$ as above. Set $\widetilde{k}_{\overline{i+1}} = k_{\overline{i+1}}$, $\widetilde{k}_{\overline{i}} = k_{\overline{i}}$, $\widetilde{k}_{i+1} = \lambda_{\overline{i}} - \lambda_{\overline{i+1}} - k_{\overline{i}} + k_i + k_{\overline{i+1}} - 1$ and $\widetilde{k}_i = (\lambda_{\overline{i+1}} - \lambda_{\overline{i}} - k_{\overline{i+1}}) + k_{i+1} + k_{\overline{i}} + 1$. Write $\widetilde{k}_t = k_t$ for any $t \neq i, i+1, \overline{i}, \overline{i+1}$. We obtain $\widetilde{k}_i \geq 0$ and $\widetilde{k}_{i+1} \geq 0$ since $\lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - k_{\overline{i+1}}$ and $\lambda_{\overline{i+1}} - k_{\overline{i+1}} < \lambda_{\overline{i}} + k_i - k_{\overline{i}}$. Since $\widetilde{k}_{\overline{s}} = k_{\overline{s}}$ for all $s = 1, ..., n-1$, we have $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - \widetilde{k}_{\overline{s+1}}$ for any $s \in \{1, ..., n-2\}$ and $\lambda_{\overline{1}} - \widetilde{k}_{\overline{1}} \geq 0$. Moreover the assertion $\lambda_{\overline{i+1}} - \widetilde{k}_{\overline{i+1}} < \lambda_{\overline{i}} + \widetilde{k}_i - \widetilde{k}_{\overline{i}}$ holds since it is equivalent to $0 < k_{i+1} + 1$. Finally $\widetilde{k}_{\overline{1}} + \cdots + \widetilde{k}_{\overline{n-1}} + \widetilde{k}_1 + \cdots + \widetilde{k}_{n-1} = k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1}$ and for any $s \in \{1, ..., n-2\}$

$$\widetilde{\gamma}_{\overline{s}} = \lambda_{\overline{s}} + \widetilde{k}_s - \widetilde{k}_{\overline{s}}.$$

Denote by E_a, E_d the sets of multi-indices $(k_{\overline{1}}, ..., k_{\overline{n-1}}, k_1, ..., k_{n-1})$ such that $k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r$ and satisfying respectively the assertions (a), (d). Let f be the map defined on $E_a \cup E_d$ by

$$f(\gamma) = \widetilde{\gamma}.$$

Then by the above arguments f is a bijection which verifies $f(E_a) = E_a$ and $f(E_d) = E_d$. Now the pairing $\gamma \longleftrightarrow \widetilde{\gamma}$ provides the cancellation of all the s_γ with $\gamma = (\lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, ..., \lambda_{\overline{1}} + k_1 - k_{\overline{1}})$ such that $(k_{\overline{1}}, ..., k_{\overline{n}}, k_1, ..., k_n) \in E_a \cup E_d$ appearing in S_1. Indeed $s_{(\mu_{\overline{n}}+R,\gamma)} = -s_{(\mu_{\overline{n}}+R,\widetilde{\gamma})}$. We obtain similarly the cancellation of all the s_γ such that γ verifies the assertions (a) or (d) appearing in S_2.
Now write E_b (resp. E_c) for the set of multi-indices $(k_{\overline{1}}, ..., k_{\overline{n-1}}, k_1, ..., k_{n-1})$ such that $k_{\overline{1}} + \cdots + k_{\overline{n-1}} + k_1 + \cdots + k_{n-1} = r$ (resp. $r-1$) and satisfying assertion (b) (resp. (c)). Let e be the map defined on $E_b \cup E_c$ by

$$e(\gamma) = \widetilde{\gamma}.$$

Then e is a bijection which verifies $e(E_b) = E_c$ and $e(E_c) = E_b$ and the s_γ such that γ verifies the assertions (b) or (c) cancel in $S_{R,r}$. Finally by Corollary 3.1.2 and Remark (iii) following Corollary 3.1.4 we obtain

$$S_{R,r} = \sum_{\gamma \in P^+_{B_{n-1}}, \lambda \in (\gamma \otimes r)_{B_{n-1}}} b^\lambda_{\gamma,r} s_{(\mu_{\overline{n}}+R,\gamma)}.$$

Note that this equality is also true when $R = r = 0$ if we set $S_{0,0} = s_{(\mu_{\overline{n}},\lambda)}$. Thus we have

$$Q_\mu = \sum_{R=0}^{+\infty} \sum_{\substack{0 \leq r \leq R \\ r \equiv R \bmod 2}} q^R \sum_{\gamma \in P^+_{B_{n-1}}, \lambda \in (\gamma \otimes r)_{B_{n-1}}} b^\lambda_{\gamma,r} K^{B_{n-1}}_{\lambda,\mu'}(q) s_{(\mu_{\overline{n}}+R,\gamma)}$$

which is equivalent to (14). So the theorem is proved. ■

Theorem 3.2.2 *Let $\mu \in P^+_{D_n}$. Then*

$$Q^{D_n}_\mu = \sum_{\gamma \in P^+_{D_{n-1}}} \sum_{R=0}^{+\infty} \sum_{r+2m=R} q^R \sum_{\lambda \in (\gamma \otimes r)_{n-1}} d^\lambda_{\gamma,r} K^{D_{n-1}}_{\lambda,\mu'}(q) s_{(\mu_{\overline{n}}+R,\gamma)}. \qquad (16)$$

Proof. Set $R_{\overline{i}} = R_{\varepsilon_{\overline{n}} - \varepsilon_{\overline{i}}}$ for $i = 1, ..., n-1$ and $R_i = R_{\varepsilon_{\overline{n}} + \varepsilon_{\overline{i}}}$ for $i = 1, ..., n$. We obtain as in proof of

Theorem 3.2.1

$$Q_\mu = \sum_{\lambda \in P^+_{D_{n-1}}} K_{\lambda,\mu'}(q) \times$$

$$\left(\sum_{R=0}^{+\infty} \sum_{\kappa_{\overline{1}}+k_{\overline{2}}+\cdots+k_{\overline{n-1}}+\kappa_1+k_2+\cdots+k_{n-1}=R} q^R (R_1)^{\kappa_1} (R_{\overline{1}})^{\kappa_{\overline{1}}} (R_2)^{k_2} (R_{\overline{2}})^{k_{\overline{2}}} \cdots (R_{n-1})^{k_{n-1}} (R_{\overline{n-1}})^{k_{\overline{n-1}}} s_{(\mu_{\overline{n}},\lambda)} \right) =$$

$$\sum_{R=0}^{+\infty} \sum_{\lambda \in P^+_{D_{n-1}}} q^R K_{\lambda,\mu'}(q) \sum_{\kappa_{\overline{1}}+k_{\overline{2}}+\cdots+k_{\overline{n-1}}+\kappa_1+k_2+\cdots+k_{n-1}=R} s_{(\mu_{\overline{n}}+R, \lambda_{\overline{n-1}}+k_{n-1}-k_{\overline{n-1}}, \cdots, \lambda_{\overline{2}}+k_2-k_{\overline{2}}, \lambda_{\overline{1}}+\kappa_1-\kappa_{\overline{1}})}.$$

Fix λ, R and consider

$$S_R = \sum_{\kappa_{\overline{1}}+k_{\overline{2}}+\cdots+k_{\overline{n-1}}+\kappa_1+k_2+\cdots+k_{n-1}=R} s_{(\mu_{\overline{n}}+R, \lambda_{\overline{n-1}}+k_{n-1}-k_{\overline{n-1}}, \cdots, \lambda_{\overline{2}}+k_2-k_{\overline{2}}, \lambda_{\overline{1}}+\kappa_1-\kappa_{\overline{1}})}.$$

Set $\gamma = (\lambda_{\overline{n-1}} + k_{n-1} - k_{\overline{n-1}}, \cdots, \lambda_{\overline{2}} + k_2 - k_{\overline{2}}, \lambda_{\overline{1}} + \kappa_1 - \kappa_{\overline{1}})$.
(a) : Consider γ appearing in S_R and suppose that there exists $i \in \{2, ..., n-2\}$ such that $\lambda_{\overline{i}} > \lambda_{\overline{i+1}} - k_{\overline{i+1}}$. Then we associate a $\widetilde{\gamma}$ verifying $\lambda_{\overline{i}} > \lambda_{\overline{i+1}} - \widetilde{k}_{\overline{i+1}}$ to γ as we have done in case (a) of the above proof. This is possible since $s_i = (\overline{i+1}, \overline{i})(i, i+1) \in W_{D_n}$.
(b) : Consider γ appearing in S_R such that $\lambda_{\overline{1}} > \lambda_{\overline{2}} - k_{\overline{2}}$. We set $\widetilde{\gamma} = s_1 \circ \gamma$, $\widetilde{k}_2 = \kappa_1$, $\widetilde{\kappa}_1 = k_2$, $\widetilde{k}_{\overline{2}} = \lambda_{\overline{2}} - \lambda_{\overline{1}} + \kappa_{\overline{1}} + 1$ and $\widetilde{\kappa}_{\overline{1}} = \lambda_{\overline{1}} - \lambda_{\overline{2}} + k_{\overline{2}} - 1$. Then γ appears in S_R and verifies $\lambda_{\overline{1}} > \lambda_{\overline{2}} - \widetilde{k}_{\overline{2}}$ whatever the sign of $\lambda_{\overline{1}}$.
(c) : Consider γ appearing in S_R such that $-\lambda_{\overline{1}} > \lambda_{\overline{2}} - k_{\overline{2}}$. We set $\widetilde{\gamma} = s_0 \circ \gamma$, $\widetilde{k}_2 = \kappa_{\overline{1}}$, $\widetilde{\kappa}_{\overline{1}} = k_2$, $\widetilde{k}_{\overline{2}} = \lambda_{\overline{2}} + \lambda_{\overline{1}} + \kappa_1 + 1$ and $\widetilde{\kappa}_1 = -\lambda_{\overline{1}} - \lambda_{\overline{2}} + k_{\overline{2}} - 1$. Then γ appears in S_R and verifies $-\lambda_{\overline{1}} > \lambda_{\overline{2}} - \widetilde{k}_{\overline{2}}$ whatever the sign of $\lambda_{\overline{1}}$.
(d) : Consider γ appearing in S_R and suppose that $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - k_{\overline{s+1}}$ for any $s \in \{2, ..., n-2\}$, $\pm\lambda_{\overline{1}} > \lambda_{\overline{2}} - k_{\overline{2}}$, and there exists $i \in \{1, ..., n-2\}$ such that $\lambda_{\overline{i+1}} - k_{\overline{i+1}} < \lambda_{\overline{i}} + k_i - k_{\overline{i}}$. We set $\widetilde{\gamma} = s_i \circ \gamma$ and proceed as in case (d) of the above proof.
(e) : Consider γ appearing in S_R and suppose that $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - k_{\overline{s+1}}$ for any $s \in \{2, ..., n-2\}$, $\pm\lambda_{\overline{1}} > \lambda_{\overline{2}} - k_{\overline{2}}$, and $\lambda_{\overline{2}} - k_{\overline{2}} < \lambda_{\overline{1}} + \kappa_1 - \kappa_{\overline{1}}$. We set $\widetilde{\gamma} = s_1 \circ \gamma$, $\widetilde{k}_{\overline{2}} = k_{\overline{2}}$, $\widetilde{\kappa}_{\overline{1}} = \kappa_{\overline{1}}$, $\widetilde{k}_2 = \lambda_{\overline{1}} - \lambda_{\overline{2}} - \kappa_{\overline{1}} + \kappa_1 - 1$ and $\widetilde{\kappa}_1 = \lambda_{\overline{2}} - \lambda_{\overline{1}} - k_{\overline{2}} + k_2 + \kappa_{\overline{1}} + 1$. Then γ appears in S_R and verifies $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - \widetilde{k}_{\overline{s+1}}$ for any $s \in \{2, ..., n-2\}$, $\pm\lambda_{\overline{1}} > \lambda_{\overline{2}} - \widetilde{k}_{\overline{2}}$, and $\lambda_{\overline{2}} - \widetilde{k}_{\overline{2}} < \lambda_{\overline{1}} + \widetilde{\kappa}_1 - \widetilde{\kappa}_{\overline{1}}$ whatever the sign of $\lambda_{\overline{1}}$.
(f) : Consider γ appearing in S_R and suppose that $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - k_{\overline{s+1}}$ for any $s \in \{2, ..., n-2\}$, $\pm\lambda_{\overline{1}} > \lambda_{\overline{2}} - k_{\overline{2}}$, and $\lambda_{\overline{2}} - k_{\overline{2}} < -\lambda_{\overline{1}} - \kappa_1 + \kappa_{\overline{1}}$. We set $\widetilde{\gamma} = s_0 \circ \gamma$, $\widetilde{k}_{\overline{2}} = k_{\overline{2}}$, $\widetilde{\kappa}_1 = \kappa_1$, $\widetilde{k}_2 = -\lambda_{\overline{1}} - \lambda_{\overline{2}} + \kappa_{\overline{1}} - \kappa_1 - 1$ and $\widetilde{\kappa}_1 = \lambda_{\overline{2}} + \lambda_{\overline{1}} - k_{\overline{2}} + k_2 + \kappa_1 + 1$. Then γ appears in S_R and verifies $\lambda_{\overline{s}} \leq \lambda_{\overline{s+1}} - \widetilde{k}_{\overline{s+1}}$ for any $s \in \{2, ..., n-2\}$, $\pm\lambda_{\overline{1}} > \lambda_{\overline{2}} - \widetilde{k}_{\overline{2}}$, and $\lambda_{\overline{2}} - \widetilde{k}_{\overline{2}} < -\lambda_{\overline{1}} + \widetilde{\kappa}_1 - \widetilde{\kappa}_{\overline{1}}$ whatever the sign of $\lambda_{\overline{1}}$.
By considering the pairing $\gamma \longleftrightarrow \widetilde{\gamma}$, the s_γ appearing in S_R cancel if they do not verify simultaneously all the following conditions

$$\begin{cases} 1 : \lambda_{\overline{1}} \leq \lambda_{\overline{2}} - \widetilde{k}_{\overline{2}} \text{ and } -\lambda_{\overline{1}} \leq \lambda_{\overline{2}} - \widetilde{k}_{\overline{2}} \\ 2 : \lambda_{\overline{i}} \leq \lambda_{\overline{i+1}} - k_{\overline{i+1}} \text{ for } i = 2, ..., n-1 \\ 3 : \lambda_{\overline{i+1}} - k_{\overline{i+1}} \geq \lambda_{\overline{i}} + k_i - k_{\overline{i}} \text{ for } i = 2, ..., n-2 \\ 4 : \lambda_{\overline{2}} - k_{\overline{2}} \leq \lambda_{\overline{1}} + \kappa_1 - \kappa_{\overline{1}} \text{ and } \lambda_{\overline{2}} - k_{\overline{2}} \leq -\lambda_{\overline{1}} + \kappa_1 - \kappa_{\overline{1}} \end{cases}. \tag{17}$$

Note that conditions $1, 2$ and 3 are precisely conditions (i), (ii) and (iii) of Corollary 3.1.4. Let E_R be the set multi-indices $M = (\kappa_{\overline{1}}, ..., k_{\overline{n}}, \kappa_1, ..., k_n)$ such that $\kappa_{\overline{1}} + \cdots + k_{\overline{n-1}} + \kappa_1 + \cdots + k_{n-1} = R$ and satisfying (17). We can write $S = \sum_{M \in E_R} s_{(\mu_{\overline{n}}+R, \gamma_M)}$. Set $E_R^- = \{M \in E_R, \kappa_1 - \kappa_{\overline{1}} \leq 0\}$ and $E_R^+ = \{M \in E_R, \kappa_1 - \kappa_{\overline{1}} > 0\}$.
Let m be an integer such that $0 \leq m \leq R/2$. Set $r = R - 2m$. Consider the multi-indices $M \in E_R^-$ such that $\kappa_1 = m$. Set $k_{\overline{1}} = \kappa_{\overline{1}} - m = \kappa_{\overline{1}} - \kappa_1$. If $\gamma_{\overline{1}} = \lambda_{\overline{1}} - k_{\overline{1}} \geq 0$ (resp. $\gamma_{\overline{1}} < 0$) then condition 4 of (17) is equivalent to condition (iv, (a)) of Corollary 3.1.4 (resp. to condition (iv, (d)). Moreover $k_{\overline{1}} + \sum_{2 \leq i \leq n} (k_{\overline{i}} + k_i) = r$. Write $B^-(r)$ for the sub-graph of $B^{D_{n-1}}(r)$ defined by the vertices which does not contain any letter 1. Set $B(\gamma) \otimes B^-(r) = \bigcup_{\lambda \in P^+_{D_{n-1}}} B(\lambda)^{d^{\lambda,-}_{\gamma,r}}$ and $(\gamma \otimes r)^-_{D_{n-1}} = \{\lambda \in P^+_{D_n}, d^{\lambda,-}_{\gamma,r} \neq 0\}$. By Remark (iii) following Corollary

3.1.4 we know that $\gamma \in P \in P_{D_{n-1}}$, so we obtain

$$\sum_{M\in E_R^-,\kappa_1=m} s_{(\mu_{\overline{n}}+R,\gamma_M)} = \sum_{\gamma\in P_{D_{n-1}},\lambda\in(\gamma\otimes r)^-_{D_{n-1}}} d^{\lambda,-}_{\gamma,r}\, s_{(\mu_{\overline{n}}+R,\gamma)}.$$

Now consider the multi-indices $M \in E_R^+$ such that $\kappa_{\overline{1}} = m$. Set $r = R-2m$ and $B(\gamma)\otimes B^+(r) = \bigcup_{\lambda\in P^+_{D_{n-1}}} B(\lambda)^{d^{\lambda,+}_{\gamma,r}}$ where $B^+(r)$ is the sub-graph of $B^{D_{n-1}}(r)$ defined by the vertices which does not contain any letter $\overline{1}$. Write $(\gamma\otimes r)^+_{D_{n-1}} = \{\lambda \in P^+_{D_n},\, d^{\lambda,+}_{\gamma,r} \neq 0\}$. We obtain similarly

$$\sum_{M\in E_R^+,\kappa_{\overline{1}}=m} s_{(\mu_{\overline{n}}+R,\gamma_M)} = \sum_{\gamma\in P_{D_{n-1}},\lambda\in(\gamma\otimes r)^+_{D_{n-1}}} d^{\lambda,+}_{\gamma,r}\, s_{(\mu_{\overline{n}}+R,\gamma)}.$$

Finally

$$S = \sum_{r+2m=R}\ \sum_{\gamma\in P_{D_{n-1}},,\lambda\in(\gamma\otimes r)^-_{D_{n-1}}\cup(\gamma\otimes r)^-_{D_{n-1}}} (d^{\lambda,-}_{\gamma,r}+d^{\lambda,+}_{\gamma,r}) s_{(\mu_{\overline{n}}+R,\gamma)} = \sum_{r+2m=R}\ \sum_{\gamma\in P_{D_{n-1}},,\lambda\in(\gamma\otimes r)_{D_{n-1}}} d^{\lambda}_{\gamma,r} s_{(\mu_{\overline{n}}+R,\gamma)}$$

since $(\gamma\otimes r)_{D_{n-1}}$ is the disjoint union of $(\gamma\otimes r)^-_{D_{n-1}}$ and $(\gamma\otimes r)^+_{D_{n-1}}$. So the theorem is proved. ■

Consider ν, μ two generalized partitions of length n. Write p for the lowest integer in $\{1,...,n\}$ such that $\nu_{\overline{p}} + p - \mu_{\overline{n}} - n \geq 0$. For any $k \in \{p, p+1, ..., n\}$ let σ_k be the signed permutation defined by

$$\sigma_k(i) = \begin{cases} i+1 \text{ if } k \leq i \leq n-1 \\ i \text{ if } 1 \leq i \leq k-1 \\ k \text{ if } i = n \end{cases}.$$

Note that $(-1)^{l_B(\sigma_k)} = (-1)^{l_D(\sigma_k)} = (-1)^{n-k}$. Let γ_k be the generalized partition of length $n-1$

$$\gamma_k = (\nu_{\overline{n}}+1, \nu_{\overline{n-1}}+1, ..., \nu_{\overline{k+1}}+1, \nu_{\overline{k-1}}, ..., \nu_{\overline{1}}).$$

Finally set $R_k = \nu_{\overline{k}} + k - \mu_{\overline{n}} - n$.
From the above recurrence formulas it is possible to express any Kostka-Foulkes polynomial $K_{\nu,\mu}(q)$ associated to a classical root system of rank n in terms of Kostka-Foulkes polynomials associated to the corresponding root system of rank $n-1$.

Theorem 3.2.3 *With the above notation we have*

$$\text{(i)} : K^{B_n}_{\nu,\mu}(q) = \sum_{k=p}^{n} (-1)^{n-k} \times q^{R_k} \times \sum_{r+2m=R_k}\ \sum_{\lambda\in(\gamma_r\otimes r)_{B_{n-1}}} b^{\lambda}_{\gamma_r,r} K^{B_{n-1}}_{\lambda,\mu'}(q),$$

$$\text{(ii)} : K^{C_n}_{\nu,\mu}(q) = \sum_{k=p}^{n} (-1)^{n-k} \times \sum_{r+2m=R_k}\ \sum_{\lambda\in(\gamma_r\otimes r)_{C_{n-1}}} q^{R_k-m} \times c^{\lambda}_{\gamma_r,r} K^{C_{n-1}}_{\lambda,\mu'}(q)$$

$$\text{(iii)} : K^{D_n}_{\nu,\mu}(q) = \sum_{k=p}^{n} (-1)^{n-k} \times q^{R_k} \times \sum_{r+2m=R_k}\ \sum_{\lambda\in(\gamma_r\otimes r)_{D_{n-1}}} d^{\lambda}_{\gamma_r,r} K^{D_{n-1}}_{\lambda,\mu'}(q).$$

Proof. In case (i), write E_ν for the set of pairs (γ, R) such that there exists $\sigma_{(\gamma,R)} \in W_{B_n}$ verifying $\sigma_{(\gamma,R)} \circ (\mu_{\overline{n}} + R, \gamma) = \nu$. We obtain from Theorems 2.1.2 and 3.2.1

$$K_{\nu,\mu}(q) = \sum_{(\gamma,R)\in E_\nu}\ \sum_{r+2m=R} q^R \sum_{\lambda\in(\gamma\otimes r)_{B_{n-1}}} b^{\lambda}_{\gamma,r} (-1)^{l(\sigma_{(\gamma,R)})} K_{\lambda,\mu'}(q). \tag{18}$$

Consider $(\gamma, R) \in E_\nu$. We must have

$$\sigma\left(\mu_{\overline{n}} + R + n - \frac{1}{2}, \gamma_{\overline{n-1}} + n - \frac{3}{2}, ..., \gamma_{\overline{1}} + \frac{1}{2}\right) = \left(\nu_{\overline{n}} + n - \frac{1}{2}, \nu_{\overline{n-1}} + n - \frac{3}{2}, ..., \nu_{\overline{1}} + \frac{1}{2}\right).$$

The strictly decreasing subsequence $(\gamma_{\overline{n-1}} + n - \frac{3}{2}, ..., \gamma_{\overline{1}} + \frac{1}{2})$ must be sent under the action of σ on a strictly decreasing subsequence I_γ of $(\nu_{\overline{n}} + n - \frac{1}{2}, \nu_{\overline{n-1}} + n - \frac{3}{2}, ..., \nu_{\overline{1}} + \frac{1}{2})$. These subsequences correspond to the choice of a $\nu_{\overline{k}} + \frac{2k-1}{2}$ (for the image of $\mu_{\overline{n}} + R + n - \frac{1}{2}$ under the action of σ) which does not belong to I_γ. For such a subsequence we must have $\mu_{\overline{n}} + R + n - \frac{1}{2} = \nu_{\overline{k}} + \frac{2k-1}{2}$. Since $R = \nu_{\overline{k}} + k - \mu_{\overline{n}} - n \geq 0$ this implies that $k \in \{p, ...n\}$, $R = R_k$, $\sigma = \sigma_k$ and $\gamma = \gamma_k$.
We prove (ii) and (iii) similarly. ■

4 The statistics χ_n^B, χ_n^C and χ_n^D

In this section we introduce a statistic on partition shaped Kashiwara-Nakashima's tableaux verifying

$$K_{\nu,\mu}(q) = \sum_{T \in \mathbf{T}(\lambda)_\mu} q^{\chi_n(T)}$$

when (ν, μ) satisfies restrictive conditions. Although the statistic χ_n can be regarded as a generalization of Lascoux-Schützenberger's charge for semi-standard tableaux, it does not permit to recover the Kostka-Foulkes polynomial $K_{\nu,\mu}(q)$ for any (ν, μ).

4.1 Catabolism

From Theorem 3.2.3 we derive the following lemma:

Lemma 4.1.1 *Let $\nu, \mu \in P_n^+$ be such that $\mu_{\overline{n}} \geq \nu_{\overline{n-1}}$. Set $l = \nu_{\overline{n}} - \mu_{\overline{n}} \geq 0$ (otherwise $K_{\nu,\mu}(q) = 0$). Then:*

$$\text{(i)} : K_{\nu,\mu}^{B_n}(q) = q^l \sum_{r+2m=l} \sum_{\lambda \in (\nu' \otimes r)_{B_{n-1}}} b_{\nu',r}^{\lambda} K_{\lambda,\mu'}^{B_{n-1}}(q),$$

$$\text{(ii)} : K_{\nu,\mu}^{C_n}(q) = \sum_{r+2m=l} q^{r+m} \sum_{\lambda \in (\nu' \otimes r)_{C_{n-1}}} c_{\nu',r}^{\lambda} K_{\lambda,\mu'}^{C_{n-1}}(q),$$

$$\text{(iii)} : K_{\nu,\mu}^{D_n}(q) = q^l \sum_{r+2m=l} \sum_{\lambda \in (\nu' \otimes r)_{D_{n-1}}} d_{\nu',r}^{\lambda} K_{\lambda,\mu'}^{D_{n-1}}(q).$$

Proof. Assertions (i), (ii) and (iii) follow by applying Theorem 3.2.3 with $p = n$. ■
From now ν and μ are generalized partitions with integers parts. Consider $T \in \mathbf{T}^n(\nu)_\mu$. Accordingly to Lemma 2.5.1, we can write

$$\mathrm{w}(T) \equiv_n \mathrm{w}(R) \otimes \mathrm{w}(T'). \tag{19}$$

Let R' be the row tableau obtained by erasing all the letters $\overline{n}$ and n in R. The catabolism of the tableau T is defined by

$$\mathrm{cat}(T) = P_{n-1}(\mathrm{w}(T') \otimes \mathrm{w}(R')).$$

The tableau $\mathrm{cat}(T)$ is well defined and belongs to $\mathbf{T}^{n-1}(\lambda)_{\mu'}$ where λ is the shape of $\mathrm{cat}(T)$ since T' and R' do not contain any letter $\overline{n}$ or n.
In the sequel we denote by ch_A the Lascoux-Schützenberger's charge statistic on semi-standard tableaux. Note that ch_A may be used to compute Kostka-Foulkes polynomials for the root systems $B_1 = C_1 = A_1$ and $D_3 = A_3$.

Consider $T \in \mathbf{T}^n(\nu)_\mu$. The statistics χ_n^B, χ_n^C and χ_n^D are defined recursively by:

$$\chi_n^B(T) = \begin{cases} \mathrm{ch}_A(T) \text{ if } n=1 \\ \chi_{n-1}^B(\mathrm{cat}(T)) + \nu_{\overline{n}} - \mu_{\overline{n}} \text{ otherwise} \end{cases}, \quad \chi_n^D(T) = \begin{cases} \mathrm{ch}_A(T) \text{ if } n=3 \\ \chi_{n-1}^D(\mathrm{cat}(T)) + \nu_{\overline{n}} - \mu_{\overline{n}} \text{ otherwise} \end{cases} \quad \text{and}$$

$$\chi_n^C(T) = \begin{cases} \mathrm{ch}_A(T) \text{ if } n=1 \\ \chi_{n-1}^C(\mathrm{cat}(T)) + \nu_{\overline{n}} - \mu_{\overline{n}} - m \text{ otherwise} \end{cases} \quad \text{where } m \text{ is the number of letters } n \text{ in } R.$$

Remark:
(i) : The statistics χ_n^B, χ_n^C and χ_n^D can be regarded as extensions of ch_A. More precisely we have $\chi_n^B(T) = \chi_n^C(T) = \chi_n^D(T) = \mathrm{ch}_A(T)$ for the tableaux T which contain only barred letters.
(ii) : To obtain χ_1^B, χ_1^C and χ_3^D we need to compute ch_A on tableaux which are not semi-standard. This can be done from the characterization of ch_A in terms of crystal graphs given in [10] or more directly by using the crystal graphs isomorphisms:

$$B(\Lambda_0^{B_1}) \simeq B(2\Lambda_1^{A_1}),\ B(\Lambda_0^{C_1}) \simeq B(\Lambda_1^{A_1}),\ B(\Lambda_0^{D_3}) \simeq B(\Lambda_3^{A_3}),\ B(\Lambda_1^{D_3}) \simeq B(\Lambda_1^{A_3}) \text{ and } B(\Lambda_2^{D_3}) \simeq B(\Lambda_2^{A_3}) \quad (20)$$

which permit to turn each tableau T related to types B_1, C_1 and D_3 into its corresponding tableau τ_T of type A_1 or A_3 via bumping algorithm on semi-standard tableaux.

Example 4.1.2 *Consider the tableau of type D_3 and shape $(3,2,1)$, $T = \begin{array}{|c|c|c|} \hline \overline{3} & \overline{2} & \overline{1} \\ \hline 1 & 2 & \\ \cline{1-2} \overline{1} & & \\ \cline{1-1} \end{array}$. Then $\mathrm{w}(T) = \overline{1}(\overline{2}2)(\overline{3}1\overline{1})$.*

We have $\overline{1} \in B(\Lambda_2^{D_3})$, $(\overline{2}2) \in B(\Lambda_1^{D_3} + \Lambda_0^{D_3}) \simeq B(\Lambda_1^{A_3} + \Lambda_3^{A_3})$ and $(\overline{3}1\overline{1}) \in B(2\Lambda_0^{D_3}) \simeq B(2\Lambda_3^{A_3})$. Thus the semi-standard tableau τ corresponding to T is obtained by applying the bumping algorithm to the word $w = (23)(3124)(124123)$. Finally $\tau_T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 2 & 3 & \\ \cline{1-4} 3 & 4 & 4 & & \\ \cline{1-3} \end{array}$.

4.2 Catabolism and Kostka-Foulkes polynomials

Consider $T \in \mathbf{T}^n(\nu)_\mu$ and suppose $n \geq 2$. For any integer $p \leq n$ consider the sequence of tableaux defined by $T_n = T$ and $T_k = \mathrm{cat}(T_{k+1})$ for $k = n-1, ..., p$. Denote by $v^{(k)} \in P_k^+$ the shape of T_k. Then $T_k \in \mathbf{T}^k(\nu^{(k)})_{\mu^{(k)}}$ with $\mu^{(k)} = (\mu_{\overline{k}}, ..., \mu_{\overline{1}})$.

Lemma 4.2.1 *If $\mu_{\overline{p}} \geq v_{\overline{n-1}}$ then for every $k = n, ..., p$ we have $\mu_{\overline{p}} \geq v_{\overline{k-1}}^{(k)}$.*

Proof. We proceed by induction on k. The lemma is true for $k = n$. Consider $k \in \{p+1, ..., n\}$ such that $\mu_{\overline{p}} \geq v_{\overline{k-1}}^{(k)}$. Then we must have $\nu_{\overline{k-2}}^{(k-1)} \leq \nu_{\overline{k-1}}^{(k)}$ by Lemmas 3.1.1 and 3.1.3 since the shape $\nu^{(k-1)}$ is obtained by adding or deleting boxes on distinct columns of the shape obtained by deleting the longest row of $\nu^{(k)}$. Hence $\nu_{\overline{k-2}}^{(k-1)} \leq \nu_{\overline{k-1}}^{(k)} \leq \mu_{\overline{p}}$. ∎

Proposition 4.2.2 *Consider ν, μ verifying one of the following conditions*
(i) :$\nu, \mu \in P_{B_n}^+$ $n = 1$ *or,* $n \geq 2$ *and* $\mu_{\overline{2}} \geq \nu_{\overline{n-1}}$
(ii) :$\nu, \mu \in P_{C_n}^+$ $n = 1$ *or,* $n \geq 2$ *and* $\mu_{\overline{2}} \geq \nu_{\overline{n-1}}$
(iii) :$\nu, \mu \in P_{D_n}^+$ $n = 3$ *or,* $n \geq 4$ *and* $\mu_{\overline{4}} \geq \nu_{\overline{n-1}}$
then

$$K_{\nu,\mu}(q) = \sum_{T \in \mathbf{T}^n(\nu)_\mu} q^{\chi_n(T)}. \quad (21)$$

Proof. The assertion is proved by induction on n.

Case (ii). The proposition is true for the root system $C_1 = A_1$. Now suppose that (21) is true for the root system C_{n-1} with $n \geq 2$ and consider ν, μ two partitions of length n such that $\mu_{\overline{2}} \geq \nu_{\overline{n-1}}$. Set $l = \nu_{\overline{n}} - \mu_{\overline{n}}$. From Lemma 4.1.1 (i) we obtain

$$K_{\nu,\mu}(q) = \sum_{r+2m=l} q^{r+m} \sum_{\lambda \in (\nu' \otimes r)_{C_{n-1}}} c_{\nu',r}^{\lambda} K_{\lambda,\mu'}(q)$$

since $\mu_{\overline{n}} \geq \mu_{\overline{2}} \geq \nu_{\overline{n-1}}$. Set
$$K(q) = \sum_{T \in \mathbf{T}^n(\nu)_\mu} q^{\chi_n(T)}.$$
Accordingly to Lemma 2.5.1, the reading of any $T \in \mathbf{T}^n(\nu)_\mu$ can be factorized as
$$\mathrm{w}(T) \equiv_n \mathrm{w}(R) \otimes \mathrm{w}(T').$$
Set $\mathcal{T}_m = \{T \in \mathbf{T}^n(\nu)_\mu, \mathrm{w}(R) \text{ contains } m \text{ letters } n\}$. We must have $0 \leq m \leq l/2$ since all the letters $\overline{n}$ or n of T belong to R and the number of letters $\overline{n}$ minus that of letters n in R must be equal to $\mu_{\overline{n}}$. For any $T \in \mathcal{T}_m$ we can write $\mathrm{cat}(T) = P_{n-1}(\mathrm{w}(T') \otimes \mathrm{w}(R'))$ where R' is a row tableau of length $r = l - 2m$. The first row of T contains at least $\mu_{\overline{n}}$ letters $\overline{n}$. Moreover we have $\mu_{\overline{n}} \geq \mu_{\overline{2}} \geq \nu_{\overline{n-1}}$. This means that $\{\mathrm{w}(R') \otimes \mathrm{w}(T'), T \in \mathcal{T}_m\} = (B((r)_{n-1}) \otimes B((\nu'))_{\mu'}$. Thus we have $\{\mathrm{w}(T') \otimes \mathrm{w}(R'), T \in \mathcal{T}_m\} = (B(\nu') \otimes B((r)_{n-1}))_{\mu'}$ and $\{(\mathrm{cat}(T), T \in \mathcal{T}_m\}$ is exactly the set of tableaux of shape $\lambda \in (\nu' \otimes r)_{C_{n-1}}$ and weight μ'. By lemma 4.2.1 we have $\mu_{\overline{2}} \geq \lambda_{\overline{n-2}}$ for any $\lambda \in B(\nu') \otimes B((r)_{n-1})$ when $n - 1 \geq 2$. So we can use the induction hypothesis and obtain
$$K(q) = \sum_{m=0}^{l/2} \sum_{T \in \mathcal{T}_m} q^{\chi_n(T)} = \sum_{r+2m=l} \sum_{T \in \mathcal{T}_m} q^{\chi_{n-1}(\mathrm{cat}(T))+l-m} = \sum_{r+2m=l} q^{r+m} \sum_{T \in \mathcal{T}_m} q^{\chi_{n-1}(\mathrm{cat}(T))} =$$
$$\sum_{r+2m=l} q^{r+m} \sum_{\lambda \in (\nu' \otimes r)_{C_{n-1}}} c^{\lambda}_{\nu', r} K_{\lambda, \mu'}(q) = K_{\nu,\mu}(q).$$
Assertions (i) and (iii) are proved similarly by induction on n starting respectively from $n = 1$ and $n = 3$. ■

Example 4.2.3 *Set $\nu = (4,1)$ and $\mu = (1,0)$ for type B_2. For the 5 corresponding tableaux of shape λ and weight μ we obtain:*

$$\chi_2^B\left(\begin{array}{|c|c|c|c|}\hline \bar{2} & \bar{1} & \bar{1} & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\right) = \chi_2^B(1\bar{1}\bar{1}\bar{2} \otimes 1) = \mathrm{ch}_A(1 \otimes 1\bar{1}\bar{1}) + 3 = 4 + 3 = 7,$$
$$\chi_2^B\left(\begin{array}{|c|c|c|c|}\hline \bar{2} & \bar{2} & 0 & 2 \\ \hline 0 \\ \cline{1-1}\end{array}\right) = \chi_2^B(20\bar{2}\bar{2} \otimes 0) = \mathrm{ch}_A(0 \otimes 0) + 3 = 1 + 3 = 4,$$
$$\chi_2^B\left(\begin{array}{|c|c|c|c|}\hline \bar{2} & \bar{1} & 0 & 1 \\ \hline 0 \\ \cline{1-1}\end{array}\right) = \chi_2^B(10\bar{1}\bar{2} \otimes 0) = \mathrm{ch}_A(0 \otimes 10\bar{1}) + 3 = 3 + 3 = 6,$$
$$\chi_2^B\left(\begin{array}{|c|c|c|c|}\hline \bar{2} & \bar{2} & \bar{1} & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\right) = \chi_2^B(2\bar{1}\bar{2}\bar{2} \otimes 1) = \mathrm{ch}_A(1 \otimes \bar{1}) + 3 = 2 + 3 = 5,$$
$$\chi_2^B\left(\begin{array}{|c|c|c|c|}\hline \bar{2} & \bar{1} & 1 & 1 \\ \hline \bar{1} \\ \cline{1-1}\end{array}\right) = \chi_2^B(11\bar{1}\bar{2} \otimes \bar{1}) = \mathrm{ch}_A(\bar{1} \otimes 11\bar{1}) + 3 = 2 + 3 = 5$$
$$\chi_2^B\left(\begin{array}{|c|c|c|c|}\hline \bar{2} & \bar{2} & 1 & 2 \\ \hline \bar{1} \\ \cline{1-1}\end{array}\right) = \chi_2^B(21\bar{2}\bar{2} \otimes \bar{1}) = \mathrm{ch}_A(\bar{1} \otimes 1) + 3 = 0 + 3 = 3.$$
Finally $K^{B_2}_{\nu,\mu}(q) = q^7 + q^6 + 2q^5 + q^4 + q^3$.

The following corollary makes clear $K_{\nu,\mu}(q)$ when ν is a row partition.

Corollary 4.2.4 *Let ν, μ be two partitions such that ν is a row partition and $\mu_{\overline{1}} \geq 0$. Set $h_n(\mu) = \sum_{i=1}^{n}(n-i)\mu_{\overline{i}}$. Then for any $R \in \mathbf{T}^n(\nu)_\mu$ we have*

(i) : $\chi_n^B(R) = h_n(\mu) + 2\sum_{i=1}^{n}(n-i+1)k_i$ *if* $0 \notin R$ *and* $\chi_n^B(R) = h_n(\mu) + 2\sum_{i=1}^{n}(n-i+1)k_i + n$ *otherwise,*

(ii) : $\chi_n^C(R) = h_n(\mu) + \sum_{i=1}^{n}(2(n-i)+1)k_i$

(iii) : $\chi_n^D(R) = h_n(\mu) + 2\sum_{i=2}^{n}(n-i+1)k_i$

where k_i is the number of letters i which belong to R.

Proof. We proceed by recurrence on n.
Suppose first $n=1$ for cases (i) and (ii). We deduce from proposition 2.2.1 that $K_{\nu,\mu}^{C_1}(q)=q^{\frac{\nu-\mu}{2}}$ and $K_{\nu,\mu}^{B_1}(q)=q^{\nu-\mu}$. Thus $\chi_1^C(R)=\frac{\nu-\mu}{2}=k_1$,

$$\chi_1^B(R)=\nu-\mu=\begin{cases}2k_1 \text{ if } 0\notin R\\ 2k_1+1 \text{ otherwise}\end{cases}$$

and the Corollary holds for $n=1$. The rest of the proof is similar to that of proposition 3.2.3 in [15].
Now suppose $n=3$ for case (iii). We can write

$$R=\begin{array}{|c|c|c|c|c|}\hline \overline{3}^{k_{\overline{3}}} & \overline{2}^{k_{\overline{2}}} & \overline{1}^{k_{\overline{1}}} & 2^{k_2} & 3^{k_3}\\ \hline\end{array}$$

where $\boxed{a^k}$ means that there are k boxes containing the letter a in R. Then the semi-standard tableau associated to R by (20) is

$$R_A=\begin{array}{|c|c|c|c|c|}\hline 1^{k_{\overline{3}}} & 1^{k_{\overline{2}}} & 2^{k_{\overline{1}}} & 2^{k_2} & 3^{k_3}\\ \hline 2^{k_{\overline{3}}} & 3^{k_{\overline{2}}} & 3^{k_{\overline{1}}} & 4^{k_2} & 4^{k_3}\\ \hline\end{array}.$$

By using the definition of the charge for semi-standard tableaux one verifies that $\mathrm{ch}(R_A)=\mu_{\overline{2}}+2\mu_{\overline{1}}+2k_3+4k_2=\chi_3^D(R)$. Thus the corollary holds for $n=3$ and we terminate as in proof of proposition 3.2.3 in [15]. ■

Remarks:
(i) : Write (r) for the row partition whose non zero part is equal to r. From Proposition 4.2.2 and Corollary 4.2.4, we deduce that for any partition $\mu\in P_+$ we have $K_{(r),\mu}(q)=q^{h_n(\mu)}\times K_{(l),0}(q)$ with $l=r-|\mu|$. If l is even we obtain $K_{(l),0}^{B_n}(q)=q^{l/2}K_{(l),0}^{C_n}(q)$ since the row tableaux of types B_n and C_n are then identical. Moreover the map t defined from $\mathbf{T}^{B_{n-1}}((l))$ to $\mathbf{T}^{D_n}((l))$ by changing each barred letter $\overline{x}$ (resp. unbarred letter x) of R into $\overline{x+1}$ (resp. $x+1$) is a bijection. Hence we have

$$K_{(l),0}^{D_n}(q)=\sum_{R\in\mathbf{T}^{D_n}((l))_0}q^{2\sum_{i=2}^{n}(n-i+1)k_i}=\sum_{t^{-1}(R)\in\mathbf{T}^{B_{n-1}}((l))_0}q^{l+\sum_{j=1}^{n-1}2(n-1-j)k_j}=K_{(l),0}^{B_{n-1}}(q)=q^{l/2}K_{(l),0}^{C_{n-1}}(q).$$

(ii) : The statistic χ_n can not be used to compute any Kostka-Foulkes polynomial. For type C_2, $\lambda=(3,1)$ and $\mu=(0,0)$ we have $K_{\lambda,\mu}(q)=q^5+q^4+q^3$. By considering the 3 tableaux of type C_2, shape λ and weight μ we obtain

$$\chi_2^C\left(\begin{array}{|c|c|c|}\hline \overline{1} & \overline{1} & 1\\ \hline 1 \\ \cline{1-1}\end{array}\right)=5,\ \chi_2^C\left(\begin{array}{|c|c|c|}\hline \overline{2} & \overline{1} & 2\\ \hline 1 \\ \cline{1-1}\end{array}\right)=3 \text{ and } \chi_2^C\left(\begin{array}{|c|c|c|}\hline \overline{2} & 1 & 2\\ \hline \overline{1} \\ \cline{1-1}\end{array}\right)=2$$

and $K_{\lambda,\mu}(q)\neq q^5+q^3+q^2$.

4.3 Cyclage graphs for the orthogonal root systems

In [15] we have introduced a (co)-cyclage graph structure on tableaux of type C. We are going to see that such a structure also exists for the partition shaped tableaux of types B and D. For any $n\geq 1$ we embed the finite alphabets $\mathcal{A}_n^B,\mathcal{A}_n^C$ and $\mathcal{A}_n^D$ respectively into the infinite alphabets

$$\mathcal{A}_\infty^B=\{\cdots<\overline{n}<\cdots<\overline{1}<0<1<\cdots<n<\cdots\}$$
$$\mathcal{A}_\infty^C=\{\cdots<\overline{n}<\cdots<\overline{1}<1<\cdots<n<\cdots\}$$
$$\mathcal{A}_\infty^D=\{\cdots<\overline{n}<\cdots<\overline{2}<\begin{smallmatrix}\overline{1}\\ 1\end{smallmatrix}<2<\cdots<n<\cdots\}.$$

The vertices of the crystal $G_\infty^B=\bigoplus_{n\geq 0}G_n^B, G_\infty^C=\bigoplus_{n\geq 0}G_n^C$ and $G_\infty^D=\bigoplus_{n\geq 0}G_n^D$ can be regarded as the words respectively on $\mathcal{A}_\infty^B,\mathcal{A}_\infty^C$ and $\mathcal{A}_\infty^D$. The congruences obtained by identifying the vertices of G_∞^B,G_∞^C and G_∞^D equal up to the plactic relations of length 3 are respectively denoted by $\equiv_B,\equiv_C$ and $\equiv_D$. Set $\mathbf{T}^B=\bigcup_{n\geq 0}\mathbf{T}_n^B$, $\mathbf{T}^C=\bigcup_{n\geq 0}\mathbf{T}_n^C$ and $\mathbf{T}^D=\bigcup_{n\geq 0}\mathbf{T}_n^D$.

By Remark (iii) before Lemma 2.5.1, there exits a unique tableau $P(w)$ such that $w \equiv \mathrm{w}(P(w))$ computed from w without using contraction relation.
In the sequel μ is a partition with n integers parts. A tableau $T \in \mathbf{T}$ is of weight $\mathrm{wt}(T) = \mu$ if $T \in \mathbf{T}_m$ with $m \geq n$, $d_{\bar{i}} = \mu_{\bar{i}}$ for $1 \leq i \leq n$ and $d_{\bar{i}} = 0$ for $i > m$. Set $\mathbf{T}^B[\mu] = \{T \in \mathbf{T}^B$ of weight $\mu\}$, $\mathbf{T}^C[\mu] = \{T \in \mathbf{T}^C$ of weight $\mu\}$ and $\mathbf{T}^D[\mu] = \{T \in \mathbf{T}^D$ of weight $\mu\}$.
Consider $T = C_1 \cdots C_r \in \mathbf{T}_\mu$ with $r > 1$ columns. The cocyclage operation is authorized for T if T contains at least a column with a letter n or without letter $\overline{n}$. In this case, let x be the rightmost letter of the longest row of T. We can write $\mathrm{w}(T) = x\mathrm{w}(T_*)$ where $T_* \in \mathbf{T}$. Then we set

$$U(T) = P(\mathrm{w}(T_*)x).$$

This means that $U(T)$ is obtained by column inserting x in T_* without using contraction relation.
Remarks:
(i): If $\mathrm{wt}(T) = 0$ then the cocyclage operation is always authorized.
(ii): By convention there is no cocyclage operation on the columns.

We endow the set $\mathbf{T}[\mu]$ with a structure of graph by drawing an array $T \to T'$ if and only if the cocyclage operation is authorized on T and $U(T) = T'$. Write $\Gamma(T)$ for the connected component containing T.

Example 4.3.1 *For $\mu = (0, 0, 0)$ the following graphs are connected components of $\mathbf{T}^B[\mu]$:*

$$\begin{array}{|c|c|c|}\hline \bar{1} & 0 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline \bar{1} & 0 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|}\hline \bar{2} \\ \hline 0 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline \bar{2} & 0 & 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline \bar{2} & 0 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bar{3} & 3 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|}\hline \bar{3} \\ \hline 0 \\ \hline 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline \bar{3} & 0 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline \bar{3} & 0 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bar{4} & 4 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|}\hline \bar{4} \\ \hline 0 \\ \hline 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|}\hline \bar{1} \\ \hline 0 \\ \hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 0 \\ \hline\end{array}.$$

All these tableaux belong to T_3^B except $\begin{array}{|c|}\hline \bar{3} \\ \hline 0 \\ \hline 3 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline \bar{3} & 0 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \bar{4} & 4 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ *which belong to T_4^B and* $\begin{array}{|c|}\hline \bar{4} \\ \hline 0 \\ \hline 4 \\ \hline\end{array}$ *which belongs to T_5^B.*

The following proposition is proved in the same way than Proposition 4.2.2 of [15].

Proposition 4.3.2 *Let $T_0 \in \mathbf{T}[0]$ and let $T_{k+1} = U(T_k)$. Then the sequence (T_n) is finite without repetition and there exists an integer e such that T_e is a column of weight 0.*

In [15] we introduce another statistic ch_{C_n} on Kashiwara-Nakashima's tableaux of type C_n based on cocyclage operation. From $T \in \mathbf{T}^C[\mu]$ we define a finite sequence of tableaux $(T_k)_{0 \leq k \leq p}$ whose last tableau T_p is a column of weight 0. When $\mu = 0$ this sequence $(T_k)_{0 \leq k \leq p}$ is precisely that given in Proposition 4.3.2. Then the statistic ch_{C_n} is first defined on the columns of weight 0 next on the tableaux by setting

$$\mathrm{ch}_{C_n}(T) = \mathrm{ch}_{C_n}(C_T) + p.$$

We conjecture that (21) holds if we replace χ_n^C by ch_{C_n} whatever the partitions λ and μ. In particular $\mathrm{ch}_{C_n}(T) \neq \chi_n(T)$ in general.
Unfortunately such a statistic defined in the same way for computing Kostka-Foulkes polynomials can not exist for the orthogonal root systems. This can be verified by considering the case $|\lambda| = 3$, $\mu = 0$ for type B_3. Set $\lambda_1 = (3, 0, 0)$, $\lambda_2 = (2, 1, 0)$ and $\lambda_3 = (1, 1, 1)$. We have $K_{\lambda_1,0}^{B_3}(q) = q^9 + q^7 + q^5$, $K_{\lambda_2,0}^{B_3}(q) = q^8 + q^7 + q^6 + q^5 + q^4$ and $K_{\lambda_3,0}^{B_3}(q) = q^6 + q^4 + q^2$. Then it is impossible to associate a statistic ch_{B_n} to the 11 tableaux of type B_3, weight 0 and shape λ_1, λ_2 or λ_3 compatible with the cyclage graph structure given in Example 4.3.1 (that is, such that $\mathrm{ch}_{B_n}(T) = \mathrm{ch}_{B_n}(T') + 1$ if $T \to T'$) and relevant for computing the corresponding Kostka-Foulkes polynomials. The situation is similar for type D_3, $|\lambda| = 3$ and $\mu = (1, 0, 0)$.

5 Explicit formulas for $K_{\lambda,\mu}(q)$

5.1 Explicit formulas for $|\lambda| \leq 3$

In the sequel we suppose that λ is a partition such that $\lambda_{\overline{1}} \geq 0$. We give below the matrix $K(q) = (K_{\lambda,\mu}(q))$ with $|\lambda| \leq 3$ associated to each root system B_n, C_n and D_n. When $|\lambda| = |\mu|$, $K_{\lambda,\mu}(q)$ can be regarded as a Kostka-Foulkes polynomial for the root system A_{n-1}. Such polynomials have been already compute (see [19] p 329). So we only give the entries of $K(q)$ corresponding to a weight μ such that $|\mu| \leq 2$. In the following matrices we have labelled the columns by λ and the rows by μ and represent each partition by its Young diagram. The expressions for the Kostka-Foulkes polynomials are obtained by using Proposition 2.2.1, Theorem 3.2.3, Proposition 4.2.2 and Corollary 4.2.4.

5.1.1 $K(q)$-matrix for the root system B_n

	(3)	$(2,1)$	$(1,1,1)$	(2)	$(1,1)$	(1)
(2)	q^n	q^{n-1}	0	1	0	0
$(1,1)$	q^{n+1}	$q^n + q^{n-1}$	q^{n-2}	q	1	0
(1)	$q^2 \times \frac{q^{2n}-1}{q^2-1}$	$q^n + q \times \frac{q^{2n-1}-1}{q-1}$	$q \times \frac{q^{2n-2}-1}{q^2-1}$	q^n	q^{n-1}	1
$\emptyset$	$q^{n+2} \times \frac{q^{2n-1}-1}{q-1}$	$q^{n+1} \times \frac{q^{2n-1}-1}{q-1}$	$q^{n-1} \times \frac{q^{2n}-1}{q^2-1}$	$q^2 \times \frac{q^{2n}-1}{q^2-1}$	$q \times \frac{q^{2n}-1}{q^2-1}$	q^n

5.1.2 $K(q)$-matrix for the root system C_n

	(3)	$(2,1)$	$(1,1,1)$	(2)	$(1,1)$
(2)	0	0	0	1	0
$(1,1)$	0	0	0	q	1
(1)	$q \times \frac{q^{2n}-1}{q^2-1}$	$q \times \frac{q^{2n-2}-1}{q-1}$	$q^2 \times \frac{q^{2n-4}-1}{q^2-1}$	0	0
$\emptyset$	0	0	0	$q \times \frac{q^{2n}-1}{q^2-1}$	$q^2 \times \frac{q^{2n-2}-1}{q^2-1}$

5.1.3 $K(q)$-matrix for the root system D_n

	(3)	$(2,1)$	$(1,1,1)$	(2)	$(1,1)$
(2)	0	0	0	1	0
$(1,1)$	0	0	0	q	1
(1)	$q^2 \times \frac{q^{2n-2}-1}{q^2-1}$	$q^{n-1} + q \times \frac{q^{2n-3}-1}{q-1}$	$q^{n-2} + q \times \frac{q^{2n-4}-1}{q^2-1}$	0	0
$\emptyset$	0	0	0	$q^2 \times \frac{q^{2n-2}-1}{q^2-1}$	$q^{n-1} + q \times \frac{q^{2n-2}-1}{q^2-1}$

Remark: For $n \geq 4$ the partitions λ and μ in the above matrix verify $\lambda^* = \lambda$ and $\mu^* = \mu$. Hence by (5) we have $K_{\lambda,\mu}^{D_n}(q) = K_{\lambda^*,\mu^*}^{D_n}(q) = K_{\lambda^*,\mu}^{D_n}(q) = K_{\lambda,\mu^*}^{D_n}(q)$.

5.2 Explicit formulas for the root system $B_2 = C_2$ and $\mu = 0$

Note first that the roots systems B_2 and C_2 are identical. More precisely denote by Ψ the linear map

$$\Psi : \begin{cases} P_{B_2}^{+} \rightarrow P_{C_2}^{+} \\ (\lambda_{\overline{2}}, \lambda_{\overline{1}}) \longmapsto (\lambda_{\overline{2}} + \lambda_{\overline{1}}, \lambda_{\overline{2}} - \lambda_{\overline{1}}) \end{cases}.$$

Accordingly to (4), the simple roots for the roots systems B_2 and C_2 are $\alpha_0^{B_2} = \varepsilon_{\overline{1}}, \alpha_1^{B_2} = \varepsilon_{\overline{2}} - \varepsilon_{\overline{1}}$ and $\alpha_0^{C_2} = 2\varepsilon_{\overline{1}}, \alpha_1^{C_2} = \varepsilon_{\overline{2}} - \varepsilon_{\overline{1}}$. Thus we have $\Psi(\alpha_0^{B_2}) = \alpha_1^{C_2}$ and $\Psi(\alpha_1^{B_2}) = \alpha_0^{C_2}$. This implies the equality

$$K_{(\lambda,\mu)}^{B_2}(q) = K_{\Psi(\lambda,\mu)}^{C_2}(q). \tag{22}$$

So it is sufficient to explicit the Kostka-Foulkes polynomials for the root system C_2.

Proposition 5.2.1 *Let $\lambda = (\lambda_{\overline{2}}, \lambda_{\overline{1}})$ be a generalized partition of length 2.*

1. *If $\lambda \in P_{+}^{C_2}$ then*

$$K_{\lambda,0}^{C_2}(q) = \begin{cases} q^{\frac{\lambda_{\overline{2}}+\lambda_{\overline{1}}}{2}} \left(\dfrac{q^{\lambda_{\overline{1}}+2}-1}{q^2-1} + q^2 \times \dfrac{q^{\lambda_{\overline{1}}+1}-1}{q-1} \times \dfrac{q^{\lambda_{\overline{2}}-\lambda_{\overline{1}}}-1}{q^2-1} \right) & \text{if } \lambda_{\overline{2}} \text{ and } \lambda_{\overline{1}} \text{ are even} \\ q^{\frac{\lambda_{\overline{2}}+\lambda_{\overline{1}}}{2}+1} \left(\dfrac{q^{\lambda_{\overline{1}}+1}-1}{q^2-1} + q \times \dfrac{q^{\lambda_{\overline{1}}+1}-1}{q-1} \times \dfrac{q^{\lambda_{\overline{2}}-\lambda_{\overline{1}}}-1}{q^2-1} \right) & \text{if } \lambda_{\overline{2}} \text{ and } \lambda_{\overline{1}} \text{ are odd} \\ 0 \text{ otherwise.} \end{cases}.$$

2. *If $\lambda \in P_{+}^{B_2}$ then*

$$K_{\lambda,0}^{B_2}(q) = \begin{cases} q^{\lambda_{\overline{2}}} \left(\dfrac{q^{2\lambda_{\overline{1}}+2}-1}{q^2-1} + q^2 \times \dfrac{q^{2\lambda_{\overline{1}}+1}-1}{q-1} \times \dfrac{q^{\lambda_{\overline{2}}-\lambda_{\overline{1}}}-1}{q^2-1} \right) & \text{if } \lambda_{\overline{2}} + \lambda_{\overline{1}} \text{ is even} \\ q^{\lambda_{\overline{2}}+1} \times \dfrac{q^{2\lambda_{\overline{1}}+1}-1}{q-1} \times \dfrac{q^{\lambda_{\overline{2}}-\lambda_{\overline{1}}+1}-1}{q^2-1} & \text{otherwise} \end{cases}.$$

Proof. 1 : Note first that $K_{\lambda,0}^{C_2}(q) = 0$ if $|\lambda|$ is odd since all the tableaux of weight 0 and type C_2 must have a pair number of boxes. So we can suppose that $\lambda_{\overline{2}}$ and $\lambda_{\overline{1}}$ have the same parity. By Theorem 3.2.3 we must have

$$K_{\lambda,0}^{C_2}(q) = \sum_{r+2m=\lambda_{\overline{2}}} q^{r+m} \sum_{\eta \in ((\lambda_{\overline{1}}) \otimes r)_1} c_{(\lambda_{\overline{1}}),r}^{\eta} K_{\eta,0}^{C_1}(q) - \sum_{r+2m=\lambda_{\overline{1}}-1} q^{r+m} \sum_{\eta \in ((\lambda_{\overline{2}}+1) \otimes r)_1} c_{(\lambda_{\overline{2}}+1),r}^{\eta} K_{\eta,0}^{C_1}(q)$$

where by abuse of notation the second sum is equal to 0 if $\lambda_{\overline{1}} = 0$. Now the $K_{\eta,0}^{C_1}(q)$'s are Kostka-Foulkes polynomials for the root system $C_1 = A_1$ hence $K_{\eta,0}^{C_1}(q) = q^{\eta/2}$. Moreover Lemma 2.3.1 implies that

$$B(\gamma) \otimes B(r) = \bigcup_{p=0}^{\min(\gamma,r)} B(\gamma + r - 2p)$$

for any integers γ, r. We obtain

$$K_{\lambda,0}^{C_2}(q) = \sum_{r+2m=\lambda_{\overline{2}}} \sum_{p=0}^{\min(\lambda_{\overline{1}},r)} q^{r+m} \times q^{\frac{\lambda_{\overline{1}}+r}{2}-p} - \sum_{r+2m=\lambda_{\overline{1}}-1} \sum_{p=0}^{r} q^{r+m} \times q^{\frac{\lambda_{\overline{2}}+r+1}{2}-p}.$$

Indeed we have $\min(\lambda_{\overline{2}} - 1, r) = r$ in the second sum since $r \leq \lambda_{\overline{1}} - 1 < \lambda_{\overline{2}} + 1$. This can be rewritten as

$$K_{\lambda,0}^{C_2}(q) = \sum_{\substack{r=0 \\ r \equiv \lambda_{\overline{2}} \bmod 2}}^{\lambda_{\overline{1}}} \sum_{p=0}^{r} q^{r+\frac{\lambda_{\overline{2}}-r}{2}+\frac{\lambda_{\overline{1}}+r}{2}-p} + \sum_{\substack{r=\lambda_{\overline{1}}+1 \\ r \equiv \lambda_{\overline{2}} \bmod 2}}^{\lambda_{\overline{2}}} \sum_{p=0}^{\lambda_{\overline{1}}} q^{r+\frac{\lambda_{\overline{2}}-r}{2}+\frac{\lambda_{\overline{1}}+r}{2}-p} - \sum_{\substack{r=0 \\ r \equiv \lambda_{\overline{1}}-1 \bmod 2}}^{\lambda_{\overline{1}}-1} \sum_{p=0}^{r} q^{r+\frac{\lambda_{\overline{1}}-r-1}{2}+\frac{\lambda_{\overline{2}}+r+1}{2}-p}$$

$$= q^{\frac{\lambda_{\overline{2}}+\lambda_{\overline{1}}}{2}} \left(\sum_{\substack{r=0 \\ r \equiv \lambda_{\overline{2}} \bmod 2}}^{\lambda_{\overline{1}}} \sum_{p=0}^{r} q^{r-p} + \sum_{\substack{r=\lambda_{\overline{1}}+1 \\ r \equiv \lambda_{\overline{2}} \bmod 2}}^{\lambda_{\overline{2}}} \sum_{p=0}^{\lambda_{\overline{1}}} q^{r-p} - \sum_{\substack{r=0 \\ r \equiv \lambda_{\overline{1}}-1 \bmod 2}}^{\lambda_{\overline{1}}-1} \sum_{p=0}^{r} q^{r-p} \right).$$

Then the Proposition easily follows by distinguishing the two cases $\lambda_{\overline{2}}$ even and $\lambda_{\overline{2}}$ odd.
2 : This is an immediate consequence of 1 and (22). ■
Remark:
(i) : Similar formulas also exist for the root system A_2. For any partition $\lambda = (a, b, 0)$ we have

$$K_{\lambda,0}^{A_2}(q) = \begin{cases} q^{a-b} \times \dfrac{q^{a+1}-1}{q-1} \text{ if } a \geq 2b \\ q^{b} \times \dfrac{q^{a-b+1}-1}{q-1} \text{ otherwise.} \end{cases} .$$

(ii) : For a weight $\mu \neq 0$, the situation becomes more complex and simple formulas for the $K_{\lambda,\mu}(q)$ seem do not exist.

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